Scientific Report

of the candidate for the title of professor

dr hab. Jacek Hejduk associative professor UŁ

Contents

1 Scientific biography

I was born on August 24, 1955 in Wolbórz in the Łódź region.

In the years 1974–1978, I had been studied at the Faculty of Mathematics, Physics and Chemistry University of Łódź and I graduated the studies with honors in the theoretical specialization. From the third year of studies, I carried out an individual study program under the supervision of Professor W. Wilczyński.

On October 1, 1978, I started working as an assistant in the Computer Department of the Institute of Mathematics at the University of Łódź.

In 1979, I completed two semester postgraduate studies at the University of Łódź in pedagogical education for the academic teachers.

Since 1981, I continued my work as an assistant in the Chair of Real Functions.

I received the PhD degree in Mathematics on September 25, 1985 at the Faculty of Mathematics, Physics and Chemistry at the University of Łódź based on the dissertation

Topologization of Boolean quotient algebras

prepared under the supervision of prof. dr hab. Władysław Wilczyński from the University of Łódź. The reviewers were:
doc. dr hab. Zbigniew Grande from the Higher Pedagogical School in Bydgoszcz,
doc. dr hab. Mirosław Filipczak from the University of Łódź.

In 1985, I completed two-year Postgraduate Studies in English at Foreign Language

Study Center at the University of Łódź.

In 1991, I took two-week internship at the University of Tbilisi in Georgia.

In 1992, during a two-month stay in East Lansing (USA), I participated in the seminar of Professor C. Weil at Michigan State University.

I received the habilitation degree on October 7, 1998 at the Faculty of Mathematics, at the University of Łódź, based on the dissertation

Density topologies with respect to invariant σ-ideals.

The reviewers were:
prof. dr hab. Julian Musielak from the University of A. Mickiewicz in Poznań,
dr hab. Tomasz Natkaniec from the University of Gdańsk,
prof. dr hab Władysław Wilczyński from the University of Łódź.

I participated in Erasmus and Erasmus+ program at the following Universities:

- 2010, University of Joannina (Greece);
- 2013, Mersin University and Çukurova University in Adana (Turkey);
- 2014, University of Santiago de Compostela (Spain);
- 2015, University of Granada (Spain);
- 2016, Istanbul Commerce University (Turkey);
- 2017, Palermo University (Italy);
- 2017, Ankara University (Turkey);
- 2018, Lucian Blaga University of Sibiu (Romania);

In 1997, during the seminar of the branch of the Slovac Academy of Sciences in Košice I presented a talk *"On the cardinality of the sets of density points"*.

In 2011, I was invited to present a plenary lecture on the topic: *On the contribution of Wilczyński's School to the density on real line* during The 25-th Summer Conference on Real Functions (Złoty Potok, Poland).

In 2016, I was invited to present a plenary talk entitled: *A visit to topological spaces generated by lower and almost lower density operators* during The 30-th Summer Conference on Real Functions (Stara Lešna, Slovakia).

Moreover, I participated over 30 conferences and meetings, mostly international, presenting my scientific results.

1) School on Real Functions, Trnava, Czechoslovakia, 1980, talk: *"On the additivity of the sets having the Baire Property"*;

2) School on Real Functions, Dubnik, Czechoslovakia, 1988, talk: *" On the topologizations of the some family of measurable functions"*;

3) The Winter School on Analysis, Srni, Czechoslovakia, 1990, talk: *"On the Lusin theorem in the aspect of small system"*;

4) The Polish-American Workshop, Łódź, 1994, talk: *"On non Baire sets in the category bases"*;

5) XX Summer Symposium in Real Analysis, Windsor, Canada,1996,talk:*"On abstract density on the real line"*;

6) Semester on the Real Functions in the Banach Center, Warsaw, Poland 1989, talk: *"On some certain σ -ideals"*;

7) XXII Summer Symposium in Real Analysis, Łódź, 1999, co-organizator of the Conference;

8) Summer School on Real Functions, Liptovský Ján, Slovakia, 2000, talk: *"On Hashimoto topology with respect to an extension of the Lebesgue measure"*;

9) X Convergeno di Analisi Reale e Teoria della Misura CARTEMI, Ischia, Italy, 2002, talk: *"On topology generated by a fix sequence"*;

10) Summer Conference on Real Functions Theory, Stará Lesná, Slovakia, 2002, talk: *"On the topology on the real line associated with the Lebesgue measure"*;

11) Real Analysis Conference, Rowy, Poland, 2003, talk: *"On $\mathcal{I}$-density topologies with respect to a fixed sequence"*;

12) XXVII Summer Symposium in Real Analysis, Opava, Czech Republic, 2003, talk: *"On the density type topologies on the real line"*;

13) XI Convergeno di Analisi Reale e Teoria della Misura CARTEMI, Ischia, Italy, 2004, talk: *"On homeomorphisms of the density type topologies"*;

14) XVIII Summer Conference on Real Functions Theory, Stará Lesná, Slovakia, 2004, talk: *"On the homeomorphisms of the density type topologies"*;

15) XIX Summer Conference on Real Functions Theory, Rowy, Poland, 2005, talk: *"On the cardinality of the set of density points"*;

16) XX Summer Conference on Real Functions Theory, Liptovský Ján, Slovakia, 2006, talk: *"On density topologies generated by functions"*;

17) XXXI Summer Symposium in Real Analysis, Trinity College, Oxford, England, 2007, talk: *"On the density topologies with respect to functions"*;

18) XXXII Summer Symposium in Real Analysis, Chicago, USA, 2008, talk: *"On the density topologies generated by sequences of intervals"*;

19) XXIII International Summer Conference on Real Functions Theory, Niedzica, Poland, 2009, talk: *"On the abstract density topologies"*;

20) International Conference on Topology and its applications, Naftpaktos, Greece, 2010, talk: *"On the abstract density topologies"*;

21) XXIV Summer Conference on Real Functions Theory, Stará Lesná, Slovakia, 2010, talk: *"On topologies with respect to σ-ideals"*;

22) The Meeting of the Israel Mathemaical Union and the Polish Mathematiacal Society, Łód, Poland, talk:*"On some kind of the generalization of the density topologies"*;

23) XXVI Summer Conference on Real Functions Theory, Stará Lesná, Slovakia, 2012, talk: *"On the topologies in the family of sets having the Baire property"*;

24) XXVIII Summer Conference on Real Functions Theory, Stará Lesná, Slovakia, 2014, talk: *"On the semiregularization of some abstract density topologies"*;

25) XXXVIII Summer Symposium in Real Analysis, Praga, Czech Republic, 2014, talk: *"On topologies generated by sequences of intervals tending to zero"*;

26) XXIX International Summer Conference on Real Functions Theory, Niedzica, Poland, 2015, talk: *"On the pointwise desnity on the real line"*;

27) I Workshop on Real Analysis, Konopnica, Poland, 2015; talk: *"Some remarks on the topologies generated by the extension of the Lebesgue measure"*;

28) XXX International Summer Conference on Real Functions Theory, Stará Lesná, Slovakia, 2016, talk: *"On generalization of the density topology"*;

29) XL Summer Symposium in Real Analysis, Sarajewo, Bosnia and Hercegovina, 2016, talk: *"On some properties of J-density topologies and J-approximately continuous functions"*;

30) II Workshop on Real Analysis, Konopnica, Poland, 2016; talk: *"On the regularity of the some kin of the density topologies"*;

31) XXXI International Summer Conference on Real Functions Theory, Ustka, Poland, 2017, talk: *"On some properties of lower and almost lower density operators"*;

32) XLII Summer Symposium in Real Analysis, Saint Petersburg, Russia, 2018, talk: *"On strong generalized topology with respect to the outer measure"*.

In the years 1991–1994, I was the participant in the Scientific Research Committee (KBN) grant conducting by Prof. W.Wilczyński.

In the years 2010–2013, I was an executor at the National Science Center Grant (N N201547238) on "Local properties of measurable sets and Baire properties in Euclidean spaces ", whose head was Prof. W. Wilczyński from the University of Łódź.

I was the reviewer in the following mathematical journals: Demonstratio Mathematica, Folia Mathematica, Georgian Mathematical Journal, Journal of Applied Analysis, Mathematica Slovaca, Positivity, Real Analysis Exchange, Journal of Applied Mathematics and Computational Mechanics.

I received the following Rector's Awards at the University of Lodz:

1980 – for scientific, educational and organizational achievements;

1989 – 2-nd degree award for educational and organizational achievements;

1990 – 3-rd degree award for a series of publications;

1992 – 2-nd degree award for a series of publications;

2005 – 1-st degree award for educational and organizational achievements;

2008 – 1-st degree award for organizational achievements;

2015 – 1-st degree team award with dr R. Wiertelak for the series of publications from the border of Topology and Measure Theory;

I received the following decorations:

2005 – Medal of the National Education Committee;

2008 – Silver Medal for Long-term Service;

2011 – Golden Award of the University of Łódź;

I have been working at the University of Łódź since October 1, 1978.

Since 2008, I have been an associate professor at the Chair of Real Functions at the Faculty of Mathematics and Computer Science of the University of Łódź.

2 Scientific achievements

2.1 Introduction

Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}_+$ – the set of all positive real numbers, $\mathbb{N}$ – the set of all positive integers, $\mathbb{Q}$ – the set of all rational numbers, $\mathcal{L}$ – the family of all measurable sets in the Lebesgue sense on $\mathbb{R}$ and $\mathbb{L}$ – the σ-ideal of the Lebesgue null sets. If $\langle X, \tau\rangle$ is a topological space, then $\mathcal{B}(\tau)$, $\mathcal{B}a(\tau)$ denote the family of all Borel sets and the family of all sets with Baire property with respect to the topology τ, respectively. Through $\mathcal{N}(\tau)$ and $\mathbb{K}(\tau)$ will be denoted the family of all nowhere dense sets and the first category sets due to the topology τ, respectively. If τ is the natural topology on $\mathbb{R}$, denoted by τ_{nat}, the designations $\mathcal{B}$, $\mathcal{B}a$, $\mathcal{N}$ and $\mathbb{K}$ for the ones described above will be used. The symbol 2^X will be used to denote the family of all subsets of X. The characteristic function of a set $A \subset X$ will be denoted by χ_A. If $b \in \mathbb{R}$, then $bA = \{b \cdot x : x \in A\}$ and $b + A = \{b + x : x \in A\}$. The Lebesgue measure on $\mathbb{R}$ will be denoted by λ, whereas its any complete extension by μ. In the further part of the Scientific Report writing about an extension of the Lebesgue measure it will always be assumed that it is a complete measure.

If $\mathcal{S}$ is a σ-algebra of the subsets of X and $\mathcal{J} \subset \mathcal{S}$ is a proper σ-ideal, then the triple

$\langle X, \mathcal{S}, \mathcal{J} \rangle$ will be called a measurable space with the distinguished σ-ideal $\mathcal{J}$. From now on writing that $\langle X, \mathcal{S}, \mathcal{J} \rangle$ is a measurable space with the distinguished σ-ideal $\mathcal{J}$ I will always mean that $\mathcal{S}$ is a σ-algebra of subsets of X and $\mathcal{J} \subset \mathcal{S}$ is a proper σ-ideal. The fact that $A \triangle B \in \mathcal{J}$ for sets A, B will be denoted by $A \sim B$. If $\mathcal{S}$ is a σ-algebra and $\mathcal{J}$ is a proper σ-ideal of subsets of X, respectively, then the smallest σ-algebra containing the family $\mathcal{S} \cup \mathcal{J}$ will be denoted by $\mathcal{S} \triangle \mathcal{J}$. Then each set $A \in \mathcal{S} \triangle \mathcal{J}$ has the form $A = B \triangle C$, where $B \in \mathcal{S}$ and $C \in \mathcal{J}$.

If $\mathcal{W}$ is a family of subsets of X and $\Phi : \mathcal{W} \to 2^X$ is such an operator that

$$\mathcal{T}_\Phi = \{A \in \mathcal{W} : A \subset \Phi(A)\}$$

forms a topology on X, then we shall say that the topology $\mathcal{T}_\Phi$ is generated by the operator Φ. In the case, if $\langle X, \mathcal{S}, \mathcal{J} \rangle$ is a measurable space with the distinguished σ-ideal $\mathcal{J}$ and $\Phi : \mathcal{S} \to 2^X$ generates $\mathcal{T}_\Phi$ topology, then we shall say that $\mathcal{T}_\Phi$ is generated by the operator Φ on $\langle X, \mathcal{S}, \mathcal{J} \rangle$.

Let $\langle X, \mathcal{S}, \mathcal{J} \rangle$ be a measurable space with the distinguished σ-ideal $\mathcal{J}$. An operator $\Phi : \mathcal{S} \to \mathcal{S}$ is called the lower density operator on $\langle X, \mathcal{S}, \mathcal{J} \rangle$, if

1° $\Phi(\emptyset) = \emptyset$, $\Phi(X) = X$;

2° $\underset{A \in \mathcal{S}}{\forall} \underset{B \in \mathcal{S}}{\forall} \Phi(A \cap B) = \Phi(A) \cap \Phi(B)$;

3° $\underset{A \in \mathcal{S}}{\forall} \underset{B \in \mathcal{S}}{\forall} A \sim B \Rightarrow \Phi(A) = \Phi(B)$;

4° $\underset{A \in \mathcal{S}}{\forall} A \sim \Phi(A)$.

If Φ is the lower density operator on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ and the pair $\langle \mathcal{S}, \mathcal{J} \rangle$ possesses the hull property then Φ generates $\mathcal{T}_\Phi$ topology on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ (J. Lukeš, J. Malý, L. Zajíček, *Fine topology methods*, Real Analysis and Potential Theory, Lecture Notes in Math. 1189, Springer-Verlag, Berlin, 1986).

If condition **??** will be replaced with the condition

($\star$) $\underset{A \in \mathcal{S}}{\forall} \Phi(A) \setminus A \in \mathcal{J}$, then the operator Φ will be called the almost lower density operator on $\langle X, \mathcal{S}, \mathcal{J}\rangle$.

In turn, if we replace condition **??** by the weaker condition

$$(\star\star) \quad \underset{A\in\mathcal{S}}{\forall} A \subset \Phi(A) \Rightarrow \Phi(A) \setminus A \in \mathcal{J},$$

then we will obtain the operator Φ called the weak almost density operator on $\langle X, \mathcal{S}, \mathcal{J}\rangle$.

However, if we give up condition **??** in the definition of the lower density operator, then the received operator Φ will be called the semi lower density operator on $\langle X, \mathcal{S}, \mathcal{J}\rangle$.

At the beginning of the 20th century H. Lebesgue introduced the notion of the density point of a measurable set in $\mathbb{R}$. Namely, if $A \in \mathcal{L}$ and $x_0 \in \mathbb{R}$, then we say that x_0 is a density point of the set A, if

$$\lim_{h\to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1.$$

In the case, when this limit equals zero, we say that x_0 is a dispersion point of the set A.

A density point can be also defined for the measurable sets in the Euclidean space $\mathbb{R}^n$, taking into account, as the neighbourhoods, the spheres or n-dimensional rectangles.

Analyzing the above definition, one can see that the essence of a density point is to investigate how much, in the sense of measure, there is of this set of in the neighbourhood of a point x_0 with respect to the measure of this neighbourhood. From the perspective of many years one can find out how delicate and deep the concept of the density in the measure theory is.

For any set $A \in \mathcal{L}$ we can define the operator $\Phi_d : \mathcal{L} \to \mathcal{L}$ as follows:

$$\Phi_d(A) = \left\{ x \in \mathbb{R} : \lim_{h\to 0^+} \frac{\lambda(A \cap [x - h, x + h])}{2h} = 1 \right\}.$$

The operator Φ_d defined in this way is the lower density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle$, whereby the property that $\lambda(A \triangle \Phi_d(A)) = 0$ is the content of fundamental Lebesgue's Theorem on the density points saying that almost every point of a set $A \in \mathcal{L}$ is its density point and almost every point of its complement is its dispersion point.

The density point was attractively adopted by A. Denjoy in 1916 (A. Denjoy, *Sur les fonctions dérvees sommables*, Bull. Soc. Math. France 43, 161–248, 1916) with in the purpose to define the approximate continuity. Namely, a function $f : \mathbb{R} \to \mathbb{R}$ is approximately continuous at $x_0 \in \mathbb{R}$, if there is a set $A \in \mathcal{L}$ such that $x_0 \in \Phi_d(A)$ and $f_{|A \cup \{x_0\}}$ is continuous at x_0 in the ordinary sense i.e. with respect to τ_{nat} topology.

O. Haupt and C. Pauc in 1952 (O. Haupt, C. Pauc, *La topologic approximative de Denjoy envisagée comme vraie toplogic*, C. R. Acad. Sci. Paris 234, 390–392, 1952) noticed that the family

$$\mathcal{T}_d = \{A \in \mathcal{L} : A \subset \Phi_d(A)\}$$

constitutes the topology significantly stronger than topology τ_{nat}. However, only in the sixties of the XX century American mathematicians C. Goffman, C. J. Neugebauer, T. Nishiura (C. Goffman, C. J. Neugebauer, T. Nishiura, *The density topology and approximate continuity*, Duke Math. J. 28, 497–506, 1961), D. Waterman (C. Goffman, D. Waterman, *Approximately continous transformation*, PAMS 12, 116–121, 1961) and further T. D. Tall (T. D. Tall, *The density topology*, Pacific J. Math. 62, No.1, 275–284, 1976) looked more closely at the topology $\mathcal{T}_d$, called the density topology, and the functions continuous with respect to this topology. During the Real Analysis Symposium in Prague in 2015 Prof. T. Nishiura told me about an enthusiastic period of research on the density topology under the direction of C. Goffman.

It has been proved, among other things, that the continuity with respect to the density topology is identical with approximate continuity. In addition, it was shown that, in the area of separation axiom, density topology is completely regular and functions continuous with respect to this topology are functions of the first class of Baire and have the Darboux property. An important feature of this topology is the fact that the collection of connected sets in this topology is identical with the family of connected sets in the natural topology. It is worth adding, that the systematic collection of properties of the density topology includes the chapter *Density topology* in Handbook of Measure Theory (Elsevier, 2002) developed by W. Wilczyńskiego.

The topology $\mathcal{T}_d$ is a special case of the von Neumanna topology corresponding to the Lebesgue measure. It is known, that every space $\langle X, \mathcal{S}_\nu, \nu \rangle$ with a distinguished, non-zero, σ-finite and complete measure ν and σ-algebra $\mathcal{S}_\nu$ of ν-measurable sets, thanks to von Neumann-Maharam's Theorem (D. Maharam, *On a theorem of von Neumann*, Proc. Amer. Math. Soc. 9, 987–994, 1958), can be associated with topology $\mathcal{T}_\nu$, corresponding to the measure ν, that is

- $\langle X, \mathcal{T}_\nu \rangle$ is a Baire space,
- $\langle X, \mathcal{T}_\nu \rangle$ fulfills the condition c.c.c (countable chain condition),
- $\mathcal{J}_\nu = \mathbb{K}(\mathcal{T}_\nu)$, where $\mathcal{J}_\nu = \{A \in \mathcal{S}_\nu : \nu(A) = 0\}$,
- $\mathcal{B}a(\mathcal{T}_\nu) = \mathcal{S}_\nu$.

Returning to the concept of a density point on $\mathbb{R}$ one can observe the attractive property, that condition

$$\lim_{h \to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1$$

is equivalent to the condition, that

$$\lim_{n \to \infty} \frac{\lambda(A \cap [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}])}{\frac{2}{n}} = 1,$$

which in turn is equivalent to the convergence with respect to the Lebesgue measure of the sequence $\{\chi_{n(A - x_0) \cap [-1,1]}\}_{n \in \mathbb{N}}$ to the function $\chi_{[-1,1]}$. This convergence, based on Riesz's Theorem, can be described by convergence almost everywhere. Exactly, it means that for every increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ of positive integers we can choose a subsequence $\{n_{k_j}\}_{j \in \mathbb{N}}$ such that

$$\chi_{n_{k_j}(A - x_0) \cap [-1,1]} \xrightarrow[j \to \infty]{} \chi_{[-1,1]} \text{ almost everywhere.}$$

This observation allowed the Authors of the paper W. Poreda, E. Wagner-Bojakowska, W. Wilczyński, *A category analogue of the density topology* (Fund. Math. 125, 167–173, 1985) to introduce the category concept of the density points.

We say that a point $x_0 \in \mathbb{R}$ is an $\mathcal{I}$-density point of a set $A \in \mathcal{B}a$, if the sequence $\{\chi_{n(A-x_0)\cap[-1,1]}\}_{n\in\mathbb{N}}$ is convergent to the function $\chi_{[-1,1]}$ with respect to the σ-ideal $\mathbb{K}$. It means that for every increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers, there exists a subsequence $\{n_{k_j}\}_{j\in\mathbb{N}}$, such that

$$\chi_{n_{k_j}(A-x_0)\cap[-1,1]} \underset{j\to\infty}{\longrightarrow} \chi_{[-1,1]} \text{ except a set of the first category.}$$

The idea of such a concept of convergence with respect to any σ-ideal, which used the reversed Riesz's Theorem, was presented by E. Wagner at the paper *Sequences of measurable functions* (Fund. Math. 112 , 89–102, 1981).

Let us for any set $A \in \mathcal{B}a$ define

$$\Phi_{\mathcal{I}}(A) = \{x \in \mathbb{R} : x \text{ is an } \mathcal{I}\text{-density point of } A\}.$$

Then we get the lower density operator on $\langle \mathbb{R}, \mathcal{B}a, \mathbb{K} \rangle$. Due to the fact the hull property for the pair $\langle \mathcal{B}a, \mathbb{K} \rangle$, the family

$$\mathcal{T}_{\mathcal{I}} = \{A \in \mathcal{B}a : A \subset \Phi_{\mathcal{I}}(A)\}$$

is a topology on $\mathbb{R}$. Moreover, this topology is stronger than the natural topology. This topology is known as the category analogue of the density topology and it was called the $\mathcal{I}$-density topology. Research on the $\mathcal{I}$-density topology and continuous functions with respect to the $\mathcal{I}$-density topology were the source of many papers in the team supervised by Professor W. Wilczyński. Furthermore K. Ciesielski, L. Larson and K. Ostaszewski devoted to this topic the monograph *$\mathcal{I}$-density continuous functions* (Mem. Amer. Math. Soc 515, 1994) to this topic.

The most important are included in the following paragraphs:

- *On the density with respect the invaiant σ-ideals*
- *On the topologies generated by operators*
- *On an abstract density topologies*
- *The $\langle s \rangle$-dnsity and f-density topology*

- *On the density topologies associated with an extension of the Lebesgue measure*

- *On the density topologies generated by the sequences of closed intervals convergent to zero*

- *Semiregularization*

- *On families of the lower density,almost lower density and semi lower density operators*

2.2 On the density with respect to the invariant σ-ideals

The concept of the $\mathcal{J}$-density was considered in paper [?] containing the results of the habilitation thesis. This concept is related to a σ-algebra $\mathcal{S} \subset 2^{\mathbb{R}}$ and a proper σ-ideal $\mathcal{J} \subset \mathcal{S}$ invariant due to linear operations $nx + b$, where $n \in \mathbb{N}$ and $b \in \mathbb{R}$. If a σ-algebra $\mathcal{S} \subset 2^{\mathbb{R}}$ and a proper σ-ideal $\mathcal{J} \subset \mathcal{S}$ are invariant in the above sense, then we say briefly that the pair $\langle \mathcal{S}, \mathcal{J} \rangle$ is invariant.

We shall say that $x_0 \in \mathbb{R}$ is a $\mathcal{J}$-density point of a set $A \in \mathcal{S}$, if the sequence $\{\chi_{n(A-x_0)\cap[-1,1]}\}_{n\in\mathbb{N}}$ is convergent to the function $\chi_{[-1,1]}$ with respect to the σ-ideal $\mathcal{J}$. It means that for an arbitrary increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers, there exists a subsequence $\{n_{k_j}\}_{j\in\mathbb{N}}$ such that

$$\chi_{n_{k_j}(A-x_0)\cap[-1,1]} \underset{j\to\infty}{\longrightarrow} \chi_{[-1,1]} \ \mathcal{J}\text{-almost everywhere,}$$

that is apart from a set belonging to σ-ideal $\mathcal{J}$.

Specifying for any $A \in \mathcal{S}$

$$\Phi_{\mathcal{J}}(A) = \{x \in \mathbb{R} : x \text{ is a } \mathcal{J}\text{-density point of } A\},$$

we get that $\Phi_{\mathcal{J}} : \mathcal{S} \to 2^X$ is the semi lower density operator on $\langle X, \mathcal{S}, \mathcal{J} \rangle$. Defining family

$$\mathcal{T}_{\mathcal{J}} = \{A \in \mathcal{S} : A \subset \Phi_{\mathcal{J}}(A)\},$$

we perceive $\mathcal{T}_{\mathcal{J}}$ as a topology if and only if $\mathcal{T}_{\mathcal{J}}$ is closed due to any sums. Observation in Theorem 2.9 at paper [?] shows that if $\mathcal{T}_{\mathcal{J}}$ is a topology, then $\text{card}(\mathcal{S}) = 2^{\mathfrak{c}}$. So that the family $\mathcal{T}_{\mathcal{J}_\omega}$ associated with the couple $\langle \mathcal{B}, \mathcal{J}_\omega \rangle$, where $\mathcal{J}_\omega$ is the σ-ideal of countable subsets of $\mathbb{R}$, is not a topology. Of course, the family $\mathcal{T}_{\mathcal{J}}$ associated with the pair $\langle 2^{\mathbb{R}}, \mathcal{J}\rangle$ for any invariant σ-ideal $\mathcal{J} \subset 2^{\mathbb{R}}$ is a topology. So I have made an observation that for any invariant σ-ideal $\mathcal{J} \subset 2^{\mathbb{R}}$ there is the smallest σ-algebra $\mathcal{S}(\mathcal{J}) \supset \mathcal{B} \cup \mathcal{J}$ such that the pair $\langle \mathcal{S}(\mathcal{J}), \mathcal{J}\rangle$ is invariant and the family

$$\mathcal{T}_{\mathcal{J}} = \{A \in \mathcal{S}(\mathcal{J}) : A \subset \Phi_J(A)\}$$

is a topology always containing natural topology. My idea of the smallest σ-algebra is close to the cosiderations of Authors of the monograph $\mathcal{I}$*-density continuous functions* (Mem. Amer. Math. Soc 515, 1994) in paragraph 1.5. The topology described above is called the $\mathcal{J}$-density topology generated by the invariant σ-ideal $\mathcal{J}$.

It is worth emphasizing here that the operator $\Phi_{\mathcal{J}}$ related to the invariant couple $\langle \mathcal{S}, \mathcal{J}\rangle$, where $\mathcal{J} \subset \mathcal{S} \subset 2^{\mathbb{R}}$ does not have to be the lower density operator on the measurable space $\langle \mathbb{R}, \mathcal{S}, \mathcal{J}\rangle$ with the distinguished σ-ideal $\mathcal{J}$. For example, if $\mathcal{J} = \mathbb{K} \cap \mathbb{L}$, then there is a set $A \in \mathcal{B} \cap (\mathbb{L} \setminus \mathbb{K})$. Hence $\Phi_{\mathcal{J}}(A) = \emptyset$. Therefore $A \setminus \Phi_{\mathcal{J}}(A) \notin \mathbb{K} \cap \mathbb{L}$. It implies that condition **??** from the definition of the lower density operator for the pair $\langle \mathcal{B} \triangle (\mathbb{K} \cap \mathbb{L}), \mathbb{K} \cap \mathbb{L}\rangle$ has not been met. At the same time the family

$$\mathcal{T}_{\mathbb{K}\cap\mathbb{L}} = \{A \in \mathcal{B} \triangle (\mathbb{K} \cap \mathbb{L}) : A \subset \Phi_{\mathbb{K}\cap\mathbb{L}}(A)\} = \mathcal{T}_d \cap \mathcal{T}_{\mathcal{I}}$$

is a topology.

In [?] I presented the results regarding the properties of the $\mathcal{J}$-density topology and the σ-algebra $\mathcal{S}(\mathcal{J})$. I noticed that the condition $\mathcal{B} \triangle \mathcal{J} \subset \mathcal{S}(\mathcal{J}) \subset 2^{\mathbb{R}}$ is always satisfied and in some case we get that $\mathcal{S}(\mathcal{J}) = \mathcal{B} \triangle \mathcal{J}$. It is true, if $\mathcal{J} = \mathbb{L}$ or $\mathcal{J} = \mathbb{K}$, or $\mathcal{J} = \mathbb{K} \cap \mathbb{L}$. In addition, I noticed that many properties can be described for invariant σ-ideals controlled by the measure and the category, which it means that $\mathcal{J} \subset \mathbb{L}$ or $\mathbb{L} \subset \mathcal{J}$, or $\mathcal{J} \subset \mathbb{K}$ or $\mathbb{K} \subset \mathcal{J}$. If $\mathcal{J} \supset \mathbb{L}$ or $\mathcal{J} \supset \mathbb{K}$, I have found that $\mathcal{S}(\mathcal{J}) = \mathcal{B} \triangle \mathcal{J}$ and $\Phi_{\mathcal{J}}$ is the lower density operator on $\langle \mathbb{R}, \mathcal{S}(\mathcal{J}), \mathcal{J}\rangle$. In addition $\mathcal{J} = \mathbb{K}$ if and only if $\mathcal{T}_{\mathcal{J}} = \mathcal{T}_{\mathcal{I}}$ and $\mathcal{J} = \mathbb{L}$ if and only if $\mathcal{T}_{\mathcal{J}} = \mathcal{T}_d$.

I proved in the habilitation thesis [?], that if an invariant σ-ideal $\mathcal{J}$ is controlled by the category, then the topological space $\langle \mathbb{R}, \mathcal{T}_{\mathcal{J}}\rangle$ is not regular. Moreover, if an invariant σ-ideal $\mathcal{J}$ contains $\mathbb{L}$ then the topological space $\langle \mathbb{R}, \mathcal{T}_{\mathcal{J}}\rangle$ is regular if and only if $\mathcal{J} = \mathbb{L}$.

I inferred that in the family of invariant σ-algebras on $\mathbb{R}$ and containing $\mathcal{B}$ the only σ-algebra $\mathcal{S}(\mathbb{L})$ such that $\Phi_{\mathbb{L}}$ is the lower density operator on $\langle\mathbb{R}, \mathcal{S}(\mathbb{L}), \mathbb{L}\rangle$ and the $\mathbb{L}$-density topology coincides with the topology $\mathcal{T}_d$ is the σ-algebra $\mathcal{L}$. I received an analogous result for the category. The only one σ-algebra $\mathcal{S}(\mathbb{K})$ with the lower density operator $\Phi_{\mathbb{K}}$ on $\langle\mathbb{R}, \mathcal{S}(\mathbb{K}), \mathbb{K}\rangle$ such that the $\mathbb{K}$-density topology coincides with the $\mathcal{I}$-density topology is the σ-algebra $\mathcal{B}a$.

Taking into account the result presented in paper [?], it can be noticed that if we consider the invariant σ-ideal $\mathcal{J}_\emptyset = \{\emptyset\}$, then we will get that x_0 is a $\mathcal{J}_\emptyset$-density point of $A \subset \mathbb{R}$, if the sequence $\{\chi_{n(A-x_0)\cap[-1,1]}\}_{n\in\mathbb{N}}$ is convergent everywhere to the function $\chi_{[-1,1]}$ or equivalently that

$$[-1,1] \subset \liminf_{n\to\infty} n(A - x_0).$$

A point of such density is shortly called a p-density point of A.

Putting for any $A \subset \mathbb{R}$

$$\Phi_p(A) = \{x \in \mathbb{R} : x \text{ is a } p\text{-density point of } A\}$$

we get that the family

$$\mathcal{T}_{p\mathcal{S}} = \{A \in \mathcal{S} : A \subset \Phi_p(A)\},$$

where $\mathcal{S}$ is an invariant σ-algebra, does not have to be closed for any unions. For example, in the case, if σ-algebra $\mathcal{S} = \mathcal{B}$. On the other hand, when $\mathcal{S} = \mathcal{L}$, then on the basis of property, $\Phi_p(A) \subset \Phi_d(A)$ for any $A \in \mathcal{L}$, we can deduce that the family

$$\mathcal{T}_{p\mathcal{L}} = \{A \in \mathcal{L} : A \subset \Phi_p(A)\}$$

is a topology stronger than the natural topology and coarser than the density topology.

This topology, referred as the point density topology, was the subject of the doctoral dissertation *The point density topology* by M. Górajska supervised by me in 2011. The case, where $\mathcal{S} = \mathcal{B}a$ was also investigated in this paper. Similarly, as in the case of a measure, due to the property that for any $A \in \mathcal{B}a$ we have that $\Phi_p(A) \subset \Phi_{\mathcal{I}}(A)$, we conclude, that the family

$$\mathcal{T}_{p\mathcal{B}a} = \{A \in \mathcal{B}a : A \subset \Phi_p(A)\}$$

is a topology. It is a topology stronger than the natural topology and coarser than the $\mathcal{I}$-density topology. There is also the interesting result in this dissertation stating that there exists a set $A \in \mathcal{L}$ such that $\Phi_p(A) \notin \mathcal{L}$ and there is a set $B \in \mathcal{B}a$ such that $\Phi_p(B) \notin \mathcal{B}a$.

Obviously, in the case of σ-ideal $\mathcal{J}_\emptyset$, similarly as in the general case of an invariant σ-ideal $\mathcal{J}$, we can say that there is the smallest invariant σ-algebra $\widehat{\mathcal{S}} \supset \mathcal{B}$ such that the family

$$\mathcal{T}_{p\widehat{\mathcal{S}}} = \{A \in \widehat{\mathcal{S}} : A \subset \Phi_p(A)\}$$

is a topology. Of course, $\widehat{\mathcal{S}} \neq \mathcal{L}$ and $\widehat{\mathcal{S}} \neq \mathcal{B}a$ because for the invariant σ-algebra $\mathcal{S} = \mathcal{L} \cap \mathcal{B}a$ we get that

$$\mathcal{T}_{p\mathcal{S}} = \{A \in \mathcal{S} : A \subset \Phi_p(A)\} = \mathcal{T}_{p\mathcal{L}} \cap \mathcal{T}_{p\mathcal{B}a},$$

so $\mathcal{T}_{p\mathcal{S}}$ is a topology. The form of the smallest σ-algebra $\widehat{\mathcal{S}}$ is still open.

In the paper [?] the question whether $\mathcal{T}_{p\mathcal{L}} \neq \mathcal{T}_{p\mathcal{B}a}$ was asked. In the submitted paper prepared jointly by M. Filipczak and M. Górajska, a positive response to this question has been presented. It also includes the result that the topological space $\langle \mathbb{R}, \mathcal{T}_{p\mathcal{L}} \rangle$ is not regular. In the case of an arbitrary invariant σ-algebra $\mathcal{S} \subset \mathcal{B}a$ and $\mathcal{S} \supset \mathcal{B}$ and such that the operator Φ_p generates topology $\mathcal{T}_{p\mathcal{S}} = \{A \in \mathcal{S} : A \subset \Phi_p(A)\}$ the result obtained in paper [?] decides that $\mathcal{B}a(\mathcal{T}_{p\mathcal{S}}) = \mathcal{B}a$ and $\mathbb{K}(\mathcal{T}_{p\mathcal{S}}) = \mathbb{K}$ and the space $\langle \mathbb{R}, \mathcal{T}_{p\mathcal{S}} \rangle$ is not regular.

Using different σ-algebras $\mathcal{L}$ and $\mathcal{B}a$ to describe the point density topologies $\mathcal{T}_{p\mathcal{L}}$ and $\mathcal{T}_{p\mathcal{B}a}$, respectively, increases the collection of differences between the measure and the category in the language of the p-density points. Namely, in paper [?] it was shown that, if 0 is a p-density point of a set $A \in \mathcal{B}a$, then there is $\varepsilon > 0$ such that $A \cap (-\varepsilon, \varepsilon)$ is a residual set on the interval $(-\varepsilon, \varepsilon)$. While there is a set $B \in \mathcal{L}$ such that 0 is a p-density point of the set B and the $B \cap (-\varepsilon, \varepsilon)$ is not a full measure on the interval $(-\varepsilon, \varepsilon)$ for any $\varepsilon > 0$. Withdrawing the Baire property or measurability of the set, we get an interesting result, presented in paper [?], that there is a set $E \notin \mathcal{L}$ such that the inner Lebesgue measure of the set E is zero and $0 \in \Phi_p(E)$. In the case of the category there is a set $E \notin \mathcal{B}a$ such that $0 \in \Phi_p(E)$ and the set $E \cap (-\varepsilon, \varepsilon)$ is no a residual set ion any interval $(-\varepsilon, \varepsilon)$ for $\varepsilon > 0$. The constructions of these sets involved the Hamel and Burstin bases, respectively.

It appears, that the concept of the $\mathcal{J}$-density topology can be also introduced on the plane with respect to σ-ideals invariant due to the operation $\langle x, y\rangle \to \langle nx, ny\rangle$ for $n \in \mathbb{N}$ and invariant due to the translation. Paper [?] is focused on such σ-ideals on the plane, called product σ-ideals of the measure and the category.

Let $A \subset \mathbb{R}$ and $s, t \in \mathbb{R}$. Let us denote $\mathrm{pr}_1(\langle s, t\rangle) = s$, $\mathrm{pr}_2(\langle s, t\rangle) = t$ and $A \pm \langle s, t\rangle = \{\langle x \pm s, y \pm t\rangle : \langle x, y\rangle \in A\}$ and $\langle s, t\rangle A = \{\langle sx, ty\rangle : \langle x, y\rangle \in A\}$. Let $[-1, 1]^2$ be the unit square centered at $\langle 0, 0\rangle$. Let $\mathcal{S}$ be a σ-algebra on the plane containing the Borel sets and $\mathcal{J} \subset \mathcal{S}$ be a proper σ-ideal on the plane such that $\mathcal{S}$ and $\mathcal{J}$ are invariant due to the operation $\langle x, y\rangle \to \langle nx, ny\rangle$ for $n \in \mathbb{N}$ and the translation.

The point $\langle 0, 0\rangle$ is a $\mathcal{J}$-density point of a set $A \in \mathcal{S}$ if the sequence $\{\chi_{\langle n,n\rangle A \cap [-1,1]^2}\}_{n \in \mathbb{N}}$ is convergent to the function $\chi_{[-1,1]^2}$ with respect to the σ-ideal $\mathcal{J}$. It means that for every increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers there exists a subsequence $\{n_{k_j}\}_{j\in\mathbb{N}}$ such that $\chi_{\langle n_{k_j}, n_{k_j}\rangle (A - x_0) \cap [-1,1]^2} \underset{j\to\infty}{\longrightarrow} \chi_{[-1,1]^2}$ $\mathcal{J}$-almost everywhere.

Let

$$\mathbb{K} \times \mathbb{L} = \left\{ E \subset \mathbb{R}^2 : \underset{B \in \mathcal{B}(\tau_{\mathbb{R}^2})}{\exists} \left(E \subset B \wedge \{x \in \mathbb{R} : B_x \notin \mathbb{L}\} \in \mathbb{K} \right) \right\},$$

where $\tau_{\mathbb{R}^2}$ is the natural topology in $\mathbb{R}^2$ and $B_x = \{y \in \mathbb{R} : \langle x, y\rangle \in B\}$. Similarly, the family $\mathbb{L} \times \mathbb{K}$ is defined. The families $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$ are proper and invariant σ-ideals on the plane. This definition is consistent with the fact that when we produce the families $\mathbb{L} \times \mathbb{L}$ and $\mathbb{K} \times \mathbb{K}$, then by Fubini and Kuratowski-Ulam theorem we get on the plane the family of null sets and the first category sets, respectively. In the case of σ-ideals $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$ we obtain that $\mathcal{S}(\mathbb{L} \times \mathbb{K}) = \mathcal{B}\ (\tau_{\mathbb{R}^2}) \triangle (\mathbb{K} \times \mathbb{L})$ and $\mathcal{S}(\mathbb{K} \times \mathbb{L}) = \mathcal{B}(\tau_{\mathbb{R}^2}) \triangle (\mathbb{L} \times \mathbb{K})$, respectively. It turns out that $\Phi_{\mathbb{K}\times\mathbb{L}}$ and $\Phi_{\mathbb{L}\times\mathbb{K}}$ are the lower density operators on $\langle \mathbb{R}^2, \mathcal{S}(\mathbb{K} \times \mathbb{L}), \mathbb{K} \times \mathbb{L}\rangle$ and $\langle \mathbb{R}^2, \mathcal{S}(\mathbb{L} \times \mathbb{K}), \mathbb{L} \times \mathbb{K}\rangle$, respectively. Theorem 2.3 in the paper by M. Gavalec, *Iterated product of ideals of Borel sets* (Colloq. Math. 50 no. 1, 39–52, 1985) and Proposition 8 in D. H. Fremlin, *Measure-additive covering of measurable selector* (Disertationes Math. 260, 1987) show that the pairs $\langle \mathcal{S}(\mathbb{K} \times \mathbb{L}), \mathbb{K} \times \mathbb{L}\rangle$ and $\langle \mathcal{S}(\mathbb{L} \times \mathbb{K}), \mathbb{L} \times \mathbb{K}\rangle$ meet c.c.c and therefore have the hull property. In the consequence the families:

$$\mathcal{T}_{\mathbb{K}\times\mathbb{L}} = \{A \in \mathcal{S}(\mathbb{K}\times\mathbb{L}) : A \subset \Phi_{\mathbb{K}\times\mathbb{L}}(A)\}$$

and

$$\mathcal{T}_{\mathbb{L}\times\mathbb{K}} = \{A \in \mathcal{S}(\mathbb{L}\times\mathbb{K}) : A \subset \Phi_{\mathbb{L}\times\mathbb{K}}(A)\}$$

are topologies containing $\tau_{\mathbb{R}^2}$, where these topologies are incomparable. The problem, if the spaces $\langle \mathbb{R}^2, \mathcal{T}_{\mathbb{K}\times\mathbb{L}}\rangle$ are $\langle \mathbb{R}^2, \mathcal{T}_{\mathbb{L}\times\mathbb{K}}\rangle$ homeomorphic has not been solved so far.

2.3 On the topologies generated by operators

My scientific research on density topologies in abstract measurable spaces proceeded towards weakening conditions imposed on the lower density operator Φ defined on a measurable space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ with the distinguished σ-ideal $\mathcal{J}$. According to the note in the Introduction, the lower density operator Φ on $\langle X, \mathcal{S}, \mathcal{J}\rangle$ with the assumption of the hull property for the pair $\langle \mathcal{S}, \mathcal{J}\rangle$ generates the topology $\mathcal{T}_\Phi = \{A \in \mathcal{S} : A \subset \Phi(A)\}$. One can even strengthen this claim that an operator Φ being the lower density operator on $\langle X, \mathcal{S}, \mathcal{J}\rangle$ generates the topology $\mathcal{T}_\Phi$ if and only if the pair $\langle \mathcal{S}, \mathcal{J}\rangle$ possesses the hull property (paper [?]). In the case of an operator Φ being the almost lower or weak almost lower density operator on $\langle X, \mathcal{S}, \mathcal{J}\rangle$ with the hull property for the pair $\langle \mathcal{S}, \mathcal{J}\rangle$, similarly as in the case of the lower density operator, Φ generates the topology

$$\mathcal{T}_\Phi = \{A \in \mathcal{S} : A \subset \Phi(A)\}.$$

Simultaneously, a simple example in paper [?] illustrates that generating a topology by the almost lower density operator on $\langle X, \mathcal{S}, \mathcal{J}\rangle$ does not imply the hull property for the pair $\langle \mathcal{S}, \mathcal{J}\rangle$.

In the case of the semi lower density operator on $\langle X, \mathcal{S}, \mathcal{J}\rangle$ it can be proved (Theorem 2 in paper [?]) that for a measurable space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ with the distinguished σ-ideal $\mathcal{J}$ such that $\mathcal{S}$ does not coincide with the smallest σ-algebra generated by $\mathcal{J}$ and $\mathcal{S} \neq 2^X$, and $\bigcup \mathcal{J} = X$, there is always the semi lower density operator on $\langle X, \mathcal{S}, \mathcal{J}\rangle$ not generating the topology $\mathcal{T}_\Phi$.

The research related to the semi lower density operator was continued in the paper [?] referring in some sense to a topology generated by invariant σ-ideals from Chapter **??**. However, in this case we are considering a more general situation, because in any topological space $\langle X, \tau \rangle$ with a distinguished proper σ-ideal $\mathcal{J}$ we associate the semi lower density operator $\Phi_{\mathcal{J}} : 2^X \to 2^X$ on the space $\langle X, 2^X, \mathcal{J} \rangle$ and the smallest σ-algebra $\mathcal{S}(\mathcal{J})$ such that $\tau \subset \mathcal{S}(\mathcal{J})$ and $\mathcal{J} \subset \mathcal{S}(\mathcal{J})$ and the family

$$\mathcal{T}_{\Phi_{\mathcal{J}}} = \{A \in \mathcal{S}(\mathcal{J}) : A \subset \Phi_{\mathcal{J}}(A)\}$$

forms a topology on X. In paper [?] some properties of the space $\langle X, \mathcal{T}_{\Phi_{\mathcal{J}}} \rangle$ and the form of σ-algebra $\mathcal{S}(\mathcal{J})$ and the families $\mathbb{K}(\mathcal{T}_{\Phi_{\mathcal{J}}})$, $\mathcal{B}a(\mathcal{T}_{\Phi_{\mathcal{J}}})$ are presented, particularly, in the case if σ-ideal $\mathcal{J}$ is controlled by σ-ideal $\mathbb{K}(\tau)$.

Similarly, in any topological space $\langle X, \tau \rangle$ with the distinguished σ-ideal $\mathcal{J}$, paper [?] discusses the regularity of the topological space $\langle X, \mathcal{T}_{\Phi} \rangle$, where $\mathcal{T}_{\Phi}$ is a topology generated by the semi lower density operator on $\langle X, \mathcal{B}a(\tau), \mathcal{J} \rangle$, if $\mathcal{J} \subset \mathbb{K}(\tau)$ or on the space $\langle X, \mathcal{B}a(\tau) \triangle \mathcal{J}, \mathcal{J} \rangle$, when $\mathcal{J} \supset \mathbb{K}(\tau)$.

It is worth quoting the result from the Theorem 3.1 in paper [57] that the topologies generated by any semi-lower density operators on the space $\langle \mathbb{R}, \mathcal{L}, \mathbb{L} \rangle$ or on space $\langle \mathbb{R}, \mathcal{B}a, \mathbb{K} \rangle$ are not normal. This partially refers to the problem 1.5.5 in paragraph 1.5 of the mentioned monograph *$\mathcal{I}$-density continuous functions*.

The studies on topologies generated by operators are included in paper [?]. There has been defined an operator $\Phi : \tau \to 2^X$ in a topological Baire space $\langle X, \tau \rangle$ with the properties:

i) $\Phi(\emptyset) = \emptyset$, $\Phi(X) = X$,

ii) $\underset{A \in \tau}{\forall} \; \underset{B \in \tau}{\forall} \; \Phi(A \cap B) = \Phi(A) \cap \Phi(B)$,

iii) $\underset{A \in \tau}{\forall} \; A \subset \Phi(A)$.

It is known that in a topological Baire space $\langle X, \tau \rangle$ each set $A \in \mathcal{B}a(\tau)$ has unambiguous representation $A = G(A) \triangle B$, where $G(A)$ is τ-regular open and $B \in \mathbb{K}(\tau)$. Using this representation, we can specify the operator $\Phi_r : \mathcal{B}a(\tau) \to \mathcal{B}a(\tau)$ by the formula

$$\Phi_r(A) = \Phi(G(A))$$

for any $A \in \mathcal{B}a(\tau)$. The operator Φ_r is the lower density operator on $\langle X, \mathcal{B}a(\tau), \mathbb{K}(\tau)\rangle$. Hence

$$\mathcal{T}_{\Phi_r} = \{A \in \mathcal{B}a(\tau) : A \subset \Phi_r(A)\}$$

is a topology containing τ. It is worth adding that in the case when Φ is constant, then $\mathcal{T}_{\Phi_r} = \tau \ominus \mathbb{K}(\tau)$. In paper [?] some properties of the topological space $\langle X, \mathcal{T}_{\Phi_r}\rangle$ have been examined. Among others the families of $\mathcal{T}_{\Phi_r}$-continuous functions and $\mathcal{T}_{\Phi_r}$-approximately continuous functions were investigated and it was shown that the above continuities do not have to be equivalent.

With any operator $\Phi : \tau \to 2^X$ having the properties ??)–??) we can associate be associated topologies $\mathcal{T}_{\Phi_{r'}}$ and $\mathcal{T}_{\Phi_{r''}}$, where

$$\mathcal{T}_{\Phi_{r'}} = \{A \subset X : A = W \cup B, W \in \tau, B \subset \Phi(W)\}$$

and

$$\mathcal{T}_{\Phi_{r''}} = \{A \subset X : A = W \cup B, W \in \tau, B \subset \Phi(G(W))\}.$$

Then $\mathcal{T}_{\Phi_{r'}} \subset \mathcal{T}_{\Phi_{r''}} \subset \mathcal{T}_{\Phi_r}$ and these inclusions are proper. Moreover,

$$\mathcal{T}_{\Phi_{r'}} \ominus \mathbb{K}(\tau) = \mathcal{T}_{\Phi_{r''}} \ominus \mathbb{K}(\tau) = \mathcal{T}_{\Phi_r},$$

and thus the equality of the families of continuous functions takes place

$$\mathcal{C}(\langle X, \mathcal{T}_{\Phi_{r'}}\rangle, \langle \mathbb{R}, \tau_{nat}\rangle) = \mathcal{C}(\langle X, \mathcal{T}_{\Phi_{r''}}\rangle, \langle \mathbb{R}, \tau_{nat}\rangle) = \mathcal{C}(\langle X, \mathcal{T}_{\Phi_r}\rangle, \langle \mathbb{R}, \tau_{nat}\rangle).$$

Paper [?] contains some properties of topologies generated by the almost lower density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle$. This paper shows, among others, that the semi lower density operator Φ on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle$ generates the topology $\mathcal{T}_\Phi$ if and only if Φ is the weak almost lower density operator i.e.

$$\forall_{A \in \mathcal{L}} \; (A \subset \Phi(A) \Rightarrow \Phi(A) \setminus A \in \mathbb{L}).$$

This paper shows that if $\Phi : \mathcal{L} \to 2^{\mathbb{R}}$ is the almost lower density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L} \rangle$, then there is always a family $\mathcal{R} \subset \mathcal{L}$ such that $\mathbb{L} \subset \mathcal{R}$, $\mathbb{R} \in \mathcal{R}$ and $\mathcal{R}$ is closed for finite products and an operator $\Phi' : \mathcal{R} \to \mathcal{L}$ such that Φ' is the almost lower density operator on $\langle \mathbb{R}, \mathcal{R}, \mathbb{L} \rangle$ and

$$\mathcal{T}_{\Phi'} = \{A \in \mathcal{R} : A \subset \Phi'(A)\}$$

is a topology such that $\mathcal{T}_{\Phi} = \mathcal{T}_{\Phi'}$.

In the case of any topological space $\langle X, \tau \rangle$, if Φ is the almost lower density operator on $\langle X, \mathcal{B}a(\tau), \mathbb{K}(\tau) \rangle$, where $X \notin \mathbb{K}(\tau)$, then Φ generates the topology $\mathcal{T}_{\Phi}$. Assuming that $\tau \subset \mathcal{T}_{\Phi}$ we get that Φ is always the lower density operator on $\langle X, \mathcal{B}a(\tau), \mathbb{K}(\tau) \rangle$.

2.4 On an abstract density topology

Let $\langle X, \tau \rangle$ be a topological space and $\langle X, \mathcal{S}, \mathcal{J} \rangle$ be a measurable space with the distinguished σ-ideal $\mathcal{J}$. We shall say that τ is an abstract density topology on $\langle X, \mathcal{S}, \mathcal{J} \rangle$, if there exists the lower density operator Φ on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ such that the topology $\mathcal{T}_{\Phi}$ generated by Φ coincides with τ.

The monograph by J. Lukeš, J. Malý, L. Zajíček *Fine topology methods* (Real Analysis and Potential Theory, Lecture Notes in Math. 1189, Springer-Verlag, Berlin, 1986) contains a few conditions characterizing, when a given topology τ is an abstract density topology on $\langle X, \mathcal{S}, \mathcal{J} \rangle$. One of them states that τ is an abstract density topology on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ if and only if the following conditions are satisfied:

a) $A \in \mathcal{J}$ if and only if A is τ-nowhere dense and τ-closed,

b) $\mathcal{S} = \mathcal{B}a(\tau)$.

Evidently the topology $\mathcal{T}_{\Phi}$ generated by the lower density operator Φ on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ is an abstract density topology on $\langle X, \mathcal{S}, \mathcal{J} \rangle$.

Considerations regarding the generating of a topology by different type of density operators can also be associated, as illustrated in paper [?], with the space $\langle X, \mathcal{A}, \mathcal{I} \rangle$, where $\mathcal{A}$ is an algebra and $\mathcal{I} \subset \mathcal{S}$ is a proper ideal of subsets of X.

Let Φ be the lower density operator on $\langle X, \mathcal{A}, \mathcal{I}\rangle$. By the standard way, we get that for the pair $\langle \mathcal{A}, \mathcal{I}\rangle$ possessing the hull property the family

$$\mathcal{T}_\Phi = \{A \in \mathcal{A} : A \subset \Phi(A)\}$$

is a topology with $\mathcal{N}(\mathcal{T}_\Phi) = \mathcal{I}$ and $\mathcal{A} = \mathcal{T}_\Phi \triangle \mathcal{N}(\mathcal{T}_\Phi)$. The inverse claim is even true, because the pair $\langle \mathcal{A}, \mathcal{I}\rangle = \langle \mathcal{T}_\Phi \triangle \mathcal{N}(\mathcal{T}_\Phi), \mathcal{N}(\mathcal{T}_\Phi)\rangle$ has the hull property.

Similarly, the almost lower density operator and the weak almost lower density operator on $\langle X, \mathcal{A}, \mathcal{I}\rangle$ generate a topology assuming that the pair $\langle \mathcal{A}, \mathcal{I}\rangle$ possesses the hull property.

Let $\langle X, \tau\rangle$ be a topological space. We shall say that τ is an abstract density topology on $\langle X, \mathcal{A}, \mathcal{I}\rangle$, if there exists the lower density operator Φ on $\langle X, \mathcal{A}, \mathcal{I}\rangle$ such that the topology $\mathcal{T}_\Phi$ generated by Φ coincides with τ. Paper [**?**] contains the result that a topology τ is an abstract density topology on $\langle X, \mathcal{A}, \mathcal{I}\rangle$ if and only if the following conditions are met:

a) $A \in \mathcal{I}$ if and only if A jest τ-nowhere dense and τ-closed,

b) $\mathcal{A} = \tau \triangle \mathcal{N}(\tau)$.

If the ideal $\mathcal{I}$ contains singletons, the lower density operator constructed in the proof of sufficient condition of the above property has the form:

$$\underset{A \in \mathcal{A}}{\forall} \Phi(A) = \operatorname{int}_\tau\{x \in X : x \in \operatorname{int}_\tau(A \cup \{x\})\}.$$

Paper [**?**] also contains the result that for any topological space $\langle X, \tau\rangle$ the following conditions are equivalent:

(a) every τ-nowhere dense set is τ-closed,

(b) there is exactly one pair $\langle \mathcal{A}, \mathcal{I}\rangle$ such that τ is an abstract density topology on $\langle X, \mathcal{A}, \mathcal{I}\rangle$, Paper [?] contains the result that if $\Phi : \mathcal{A} \to 2^X$ is the semi lower density operator on $\langle X, \mathcal{A}, \mathcal{I}\rangle$ and Φ generates the topology $\mathcal{T}_\Phi$, then $\mathcal{N}(\mathcal{T}_\Phi) = \mathcal{I}$ if and only if there exists an algebra $\mathcal{A}' \subset \mathcal{A}$ such that $\mathcal{I} \subset \mathcal{A}'$ and $\mathcal{T}_\Phi$ is an abstract density topology on $\langle X, \mathcal{A}', \mathcal{I}\rangle$. Another result in paper [?] is that, if $\Phi : \mathcal{A} \to 2^X$ is the semi lower density operator on $\langle X, \mathcal{A}, \mathcal{I}\rangle$ generating the topology $\mathcal{T}_\Phi$, then $\mathcal{T}_\Phi$ is an abstract density topology on $\langle X, \mathcal{A}, \mathcal{I}\rangle$ if and only if $\mathcal{N}(\mathcal{T}_\Phi) = \mathcal{I}$ and $\mathcal{A} = \mathcal{T}_\Phi \triangle \mathcal{I}$. Additionally, paper [?] contains the observation, that if $\Phi : \mathcal{S} \to 2^X$ is the almost lower density operator then there is a subfamily $\mathcal{R}$ such that $\mathcal{I} \subset \mathcal{R} \subset \mathcal{S}$, $X \in \mathcal{R}$, $\mathcal{R}$ is closed due to finite product and there is the almost lower density operator $\Phi' : \mathcal{R} \to \mathcal{S}$ generating the topology

$$\mathcal{T}_{\Phi'} = \{A \in \mathcal{R} : A \subset \Phi'(A)\} = \mathcal{T}_\Phi.$$

In the research on the abstract density topologies some duality and differences between the measure and the category can be observed. It was observed in paper [?] that in a topological Baire space $\langle X, \tau\rangle$, there is the lower density operator Φ on $\langle X, \mathcal{B}a(\tau), \mathbb{K}(\tau)\rangle$ generating the topology $\mathcal{T}_\Phi$ such that

$$\mathcal{T}_\Phi = \tau \ominus \mathbb{K}(\tau)$$

So the conclusion is that the Hashimoto type topology $\tau_{nat} \ominus \mathbb{K}(\tau)$ is an abstract density topology on $\langle X, \mathcal{B}a, \mathbb{K}\rangle$. The dual Hashimoto type topology $\tau_{nat} \ominus \mathbb{L}$ does not possess this property anymore. Namely, paper [?] decides that there does not exist a measurable space $\langle \mathbb{R}, \mathcal{S}, \mathcal{J}\rangle$ with the distinguished σ-ideal $\mathcal{J}$ such that the topology $\tau_{nat} \ominus \mathbb{L}$ is an abstract density topology on $\langle \mathbb{R}, \mathcal{S}, \mathcal{J}\rangle$. It turns out, however, that the topology $\tau_{nat} \ominus \mathbb{L}$ is the almost density topology on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle$, it means that there exists the almost lower density operator Φ on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle$ generating the topology $\mathcal{T}_\Phi$ identical with $\tau_{nat} \ominus \mathbb{L}$.

2.5 The $\langle s\rangle$-density and f-density topology

The definition of a density point of a set $A \in \mathcal{L}$ involving the sequence $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ was an inspiration in paper [?] to define a density point with respect to an unbounded and nondecreasing sequence of positive numbers $\{s_n\}_{n\in\mathbb{N}}$. The family of such sequences is denoted by $\mathbb{S}$ and a sequence $\{s_n\}_{n\in\mathbb{N}} \in \mathbb{S}$ by $\langle s\rangle$.

Let $\langle s\rangle \in \mathbb{S}$. We shall say that $x_0 \in \mathbb{R}$ is an $\langle s\rangle$-density point of a set $A \in \mathcal{L}$, if

$$\lim_{n\to\infty} \frac{\lambda(A \cap [x_0 - \frac{1}{s_n}, x_0 + \frac{1}{s_n}])}{\frac{2}{s_n}} = 1.$$

Let

$$\Phi_{\langle s\rangle}(A) = \{x \in \mathbb{R} : x \text{ is an } \langle s\rangle\text{-density point of } A\}.$$

for any $A \in \mathcal{L}$ and $\langle s\rangle \in \mathbb{S}$. Then, for every $\langle s\rangle \in \mathbb{S}$ an operator $\Phi_{\langle s\rangle} : \mathcal{L} \to \mathcal{L}$ is the lower density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle$ an finally

$$\mathcal{T}_{\langle s\rangle} = \{A \in \mathcal{L} : A \subset \Phi_{\langle s\rangle}(A)\}$$

is a topology containing τ_{nat} and it is called the $\langle s\rangle$-density topology. The interesting result of paper [?] states, that

$$\mathcal{T}_d = \mathcal{T}_{\langle s\rangle} \Leftrightarrow \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} > 0.$$

Let

$$\mathbb{S}_0 = \{\langle s\rangle \in \mathbb{S} : \lim_{n\to\infty} \frac{s_n}{s_{n+1}} = 0\}.$$

If $\langle s\rangle \in \mathbb{S}_0$, then the topology $\mathcal{T}_{\langle s\rangle}$ is the essential extension of the topology $\mathcal{T}_d$. For every $\langle s\rangle \in \mathbb{S}$ and $m \in \mathbb{R}$ such that $|m| \geq 1$ and $b \in \mathbb{R}$, $\mathcal{T}_{\langle s\rangle}$-open sets are always invariant with respect to the linear operation $x \mapsto mx + b$. However, for any $\langle s\rangle \in \mathbb{S}_0$ and $m \in \mathbb{R}$ such that $|m| < 1$ there exists a $\mathcal{T}_{\langle s\rangle}$-open set A such that $mA \notin \mathcal{T}_{\langle s\rangle}$. Moreover, if $\mathbb{T} = \{\mathcal{T}_{\langle s\rangle} : \langle s\rangle \in \mathbb{S}\}$, then $\text{card}(\mathbb{T}) = \mathfrak{c}$ and

$$\bigcap_{\langle s\rangle \in \mathbb{S}_0} \mathcal{T}_{\langle s\rangle} = \mathcal{T}_d.$$

At the same time, the family $\bigcup_{\langle s\rangle \in \mathbb{S}} \mathcal{T}_{\langle s\rangle}$ is not a topology and the smallest one containing $\bigcup_{\langle s\rangle \in \mathbb{S}} \mathcal{T}_{\langle s\rangle}$ is $2^{\mathbb{R}}$.

Paper [?] contains the result concerning the comparison of the $\langle s\rangle$-density type topologies. If

$$\langle s\rangle, \langle t\rangle \in \mathbb{S}_0 \text{ i } \lim_{n\to\infty} \tfrac{s_n}{t_n} = \alpha \in (0,\infty),$$

then

$$\mathcal{T}_{\langle s\rangle} = \mathcal{T}_{\langle t\rangle} \text{ if and only if } \alpha = 1.$$

Paper [**?**] contains the further comparison of the $\langle s\rangle$-density type topologies. This paper states, that for every sequence $\langle t\rangle \in \mathbb{S}_0$ there is a sequence $\langle s\rangle \in \mathbb{S}_0$ such that the topology $\mathcal{T}_{\langle t\rangle}$ is essentially stronger than the topology $\mathcal{T}_{\langle s\rangle}$. It implies that for every $\langle t\rangle \in \mathbb{S}$ there exists a sequence $\langle s\rangle \in \mathbb{S}$ such that the topology $\mathcal{T}_{\langle s\rangle}$ is essentially stronger than $\mathcal{T}_{\langle t\rangle}$.

It is worth adding that $\langle s\rangle$-density topologies for $\langle s\rangle \in \mathbb{S}_0$ can not be comparable. There exist sequences $\langle s\rangle, \langle t\rangle \in \mathbb{S}_0$ such that $\mathcal{T}_{\langle t\rangle} \setminus \mathcal{T}_{\langle s\rangle} \neq \emptyset$ and $\mathcal{T}_{\langle s\rangle} \setminus \mathcal{T}_{\langle t\rangle} \neq \emptyset$. The criterion of the comparison of the $\langle s\rangle$-density topologies is described by the same properties of the sequences from $\mathbb{S}$. Namely, for fix $\langle s\rangle, \langle t\rangle \in \mathbb{S}$ let us define a sequence $\{k_n\}_{n\in\mathbb{N}}$ defined by the condition that for every $n \in \mathbb{N}$ there exists the exactly one number k_n such that $t_n \in (s_{k_n}, s_{k_n+1}]$. Then

$$\mathcal{T}_{\langle s\rangle} \subset \mathcal{T}_{\langle s\rangle}$$

if and only if

$$\liminf_{i\to\infty} \frac{t_{n_i}}{s_{k_{n_i}}} < \infty \text{ or } \liminf_{i\to\infty} \frac{s_{k_{n_i}+1}}{t_{n_i}} = 1.$$

The equality $\mathcal{T}_{\langle s\rangle} = \mathcal{T}_{\langle t\rangle}$ for $\langle s\rangle, \langle t\rangle \in \mathbb{S}$ can be obtained by the comparison of the family of continuous functions $\mathcal{C}(\langle \mathbb{R}, \mathcal{T}_{\langle s\rangle}\rangle, \langle \mathbb{R}, \tau_{nat}\rangle)$ and $\mathcal{C}(\langle \mathbb{R}, \mathcal{T}_{\langle t\rangle}\rangle, \langle \mathbb{R}, \tau_{nat}\rangle)$. It holds that $\mathcal{T}_{\langle s\rangle} = \mathcal{T}_{\langle t\rangle}$ if and only if

$$\mathcal{C}(\langle \mathbb{R}, \mathcal{T}_{\langle s\rangle}\rangle, \langle \mathbb{R}, \tau_{nat}\rangle) = \mathcal{C}(\langle \mathbb{R}, \mathcal{T}_{\langle t\rangle}\rangle, \langle \mathbb{R}, \tau_{nat}\rangle),$$

which is the consequence of the fact that the topological space $\langle \mathbb{R}, \mathcal{T}_{\langle s\rangle}\rangle$ is completely regular for a fix $\langle s\rangle \in \mathbb{S}$. This result and more properties related to the $\langle s\rangle$-density topology and the family of functions continuous with respect to such a topology are contained in the doctoral dissertation of A. Loranty *On density topologies induced by the nondecreasing and unbounded sequences of positive reals* prepared under my supervision in 2005.

The considerations on $\langle s\rangle$-density topologies are involved in paper [**?**]. Its main purpose is the result that for $\langle s\rangle, \langle t\rangle \in \mathbb{S}_0$, if topologies $\mathcal{T}_{\langle s\rangle}$ and $\mathcal{T}_{\langle t\rangle}$ are not comparable then the topological spaces $\langle \mathbb{R}, \mathcal{T}_{\langle s \rangle} \rangle$ and $\langle \mathbb{R}, \mathcal{T}_{\langle t \rangle} \rangle$ are not homeomorphic. The assumption related non comparability $\mathcal{T}_{\langle s \rangle}$ and $\mathcal{T}_{\langle t \rangle}$ is essential, because for every $\langle s \rangle \in \mathbb{S}_0$ there exists $\langle t \rangle \in \mathbb{S}_0$ such that $\mathcal{T}_{\langle t \rangle} \subsetneq \mathcal{T}_{\langle s \rangle}$ and the topological spaces $\langle \mathbb{R}, \mathcal{T}_{\langle s \rangle} \rangle$ and $\langle \mathbb{R}, \mathcal{T}_{\langle t \rangle} \rangle$ are homeomorphic. Similarly, for every $\langle t \rangle \in \mathbb{S}_0$ there exists a sequence $\langle s \rangle \in \mathbb{S}_0$ such that $\mathcal{T}_{\langle t \rangle} \subsetneq \mathcal{T}_{\langle s \rangle}$ and the topological spaces $\langle \mathbb{R}, \mathcal{T}_{\langle s \rangle} \rangle$ and $\langle \mathbb{R}, \mathcal{T}_{\langle t \rangle} \rangle$ are homeomorphic. Simultaneously the result that for every $\langle s \rangle \in \mathbb{S}_0$ the topological spaces $\langle \mathbb{R}, \mathcal{T}_{\langle s \rangle} \rangle$ and $\langle \mathbb{R}, \mathcal{T}_d \rangle$ are not homeomorphic is true. Therefore the investigation of $\mathcal{T}_{\langle s \rangle}$ topology for $\langle s \rangle \in \mathbb{S}_0$ make sense. The proofs of all the above results required studying many properties. For example, to show that an $\langle s \rangle$-approximately continuous function has the Darboux property. The fact that $f : \mathbb{R} \to \mathbb{R}$ is $\langle s \rangle$-approximately continuous at $x_0 \in \mathbb{R}$ means that $x_0 \in \Phi_{\langle s \rangle}(A)$ for some $A \in \mathcal{L}$ and $f_{|A \cup \{x_0\}}$ is continuous in τ_{nat}. The property that $\mathcal{T}_{\langle s \rangle}$-connected sets coincide with τ_{nat}-connected sets was very helpful, either.

Paper [?] contains the further results related to the cardinality of homeomorphic and non homeomorphic families of $\langle s \rangle$-density topologies. For every sequence $\langle s \rangle \in \mathbb{S}$ there exists a set $\mathbb{S}_1 \subset \mathbb{S}$ continuum cardinality such that for every two different sequences $\langle u \rangle, \langle t \rangle \in \mathbb{S}_1$ we get different topologies $\mathcal{T}_{\langle u \rangle}$ and $\mathcal{T}_{\langle t \rangle}$ and for every $\langle t \rangle \in \mathbb{S}_1$ the topological spaces $\langle \mathbb{R}, \mathcal{T}_{\langle s \rangle} \rangle$ and $\langle \mathbb{R}, \mathcal{T}_{\langle t \rangle} \rangle$ are homeomorphic. From the other side, there exists a set $\mathbb{S}_2 \subset \mathbb{S}$ continuum cardinality such that for different sequences $\langle u \rangle, \langle t \rangle \in \mathbb{S}_2$ we get different topologies $\mathcal{T}_{\langle u \rangle}$ and $\mathcal{T}_{\langle t \rangle}$ and for every $\langle t \rangle \in \mathbb{S}_2$ the topological spaces $\langle \mathbb{R}, \mathcal{T}_{\langle s \rangle} \rangle$ and $\langle \mathbb{R}, \mathcal{T}_{\langle t \rangle} \rangle$ are not homeomorphic. Moreover, there exists a set $\mathbb{S}_3 \subset \mathbb{S}$ continuum cardinality such that for different sequences $\langle u \rangle, \langle t \rangle \in \mathbb{S}_3$ the topological spaces $\langle \mathbb{R}, \mathcal{T}_{\langle u \rangle} \rangle$ and $\langle \mathbb{R}, \mathcal{T}_{\langle t \rangle} \rangle$ are not homeomorphic.

The analogue of the $\langle s \rangle$-density topology in the aspect of the category is contained in paper [?]. If $\langle s \rangle \in \mathbb{S}$, then for an arbitrary set $A \in \mathcal{B}a$ a point $x_0 \in \mathbb{R}$ is called an $\mathcal{I}_{\langle s \rangle}$-density point of the set A, if the sequence $\{\chi_{s_n(A-x_0)\cap[-1,1]}\}_{n\in\mathbb{N}}$ is convergent to $\chi_{[-1,1]}$ with respect to σ-ideal $\mathbb{K}$. Defining

$$\Phi_{\mathcal{I}_{\langle s \rangle}}(A) = \{x \in \mathbb{R} : x \text{ is an } \mathcal{I}_{\langle s \rangle}\text{-density point of } A\}$$

for any $A \in \mathcal{B}a$, we get that $\Phi_{\mathcal{I}_{\langle s \rangle}} : \mathcal{B}a \to \mathcal{B}a$ is the lower density operator on $\langle \mathbb{R}, \mathcal{B}a, \mathbb{K} \rangle$.

It implies that the family

$$\mathcal{T}_{\mathcal{I}_{\langle s\rangle}} = \{A \in \mathcal{B}a : A \subset \Phi_{\mathcal{I}_{\langle s\rangle}}(A)\}$$

forms a topology on $\mathbb{R}$ stronger than τ_{nat}. Moreover, for an arbitrary $\langle s\rangle \in \mathbb{S}$ we get that $\mathcal{T}_{\mathcal{I}} \subset \mathcal{T}_{\mathcal{I}_{\langle s\rangle}}$ and $\mathcal{T}_{\mathcal{I}_{\langle s\rangle}} = \mathcal{T}_{\mathcal{I}}$ if and only if $\liminf\limits_{n\to\infty} \frac{s_n}{s_{n+1}} > 0$. Similarly, as in the case of the $\langle s\rangle$-density topology with respect to a measure, we infer that

$$\bigcap_{\langle s\rangle\in\mathbb{S}_0} \mathcal{T}_{\mathcal{I}_{\langle s\rangle}} = \mathcal{T}_{\mathcal{I}}.$$

The topology $\mathcal{T}_{\mathcal{I}_{\langle s\rangle}}$ is invariant with respect to the linear transformation $mx + b$, where $m \in \mathbb{R}$ and $|m| \geq 1$, and $b \in \mathbb{R}$. However, for every $\langle s\rangle \in \mathbb{S}_0$ and $m \in \mathbb{R}$ such that $|m| < 1$, there exists a set $A \in \mathcal{T}_{\mathcal{I}_{\langle s\rangle}}$ with $mA \notin \mathcal{T}_{\mathcal{I}_{\langle s\rangle}}$.

In paper [?] the concept of density with respect to a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\limsup\limits_{x\to 0^+} \frac{x}{f(x)} > 0$ was introduced. We shall say that $x_0 \in \mathbb{R}$ is an f-density point of a set $A \in \mathcal{L}$, if

$$\lim_{h\to 0^+} \frac{\lambda(A' \cap [x_0 - h, x_0 + h])}{f(2h)} = 0.$$

We call such a point a symmetrical f-density point. If

$$\Phi^s_{\langle f\rangle}(A) = \{x \in \mathbb{R} : x \text{ is a symmetrical } f\text{-density point of } A\},$$

for $A \in \mathcal{L}$, then $\Phi^s_{\langle f\rangle} : \mathcal{L} \to 2^{\mathbb{R}}$ is the almost lower density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle$. Then the family

$$\mathcal{T}^s_{\langle f\rangle} = \{A \in \mathcal{L} : A \subset \Phi^s_{\langle f\rangle}(A)\}$$

is a topology stronger than τ_{nat}. It is the generalization of the density topology $\mathcal{T}_d$. A symmetrical f-density, further investigated in [?], is associated with an f-density have been introduced earlier by M. Filipczak and T. Filipczak in the paper *A generalization of the density topology* (M. Filipczak, T. Filipczak, *A generalization of the density topology*, Tatra Mt. Math. Publ. 34, 37–45, 2006). There has been introduced the notion of an f-density point for a function $f : \mathbb{R}_+ \to \mathbb{R}_+$, which is non-increasing, $\lim\limits_{x\to 0^+} f(x) = 0$ and $\limsup\limits_{x\to 0^+} \frac{x}{f(x)} > 0$. Then a point $x_0 \in \mathbb{R}$ is an f-density point of a set

$A \in \mathcal{L}$, if

$$\lim_{\substack{h\to 0, k\to 0\\ h\geq 0, k\geq 0\\ h+k>0}} \frac{\lambda(A' \cap [x_0 - h, x_0 + k])}{f(h+k)} = 0.$$

In the footstep of this density the Authors obtained the f-density topology

$$\mathcal{T}_{\langle f\rangle} = \{A \in \mathcal{L} : A \subset \Phi_{\langle f\rangle}(A)\},$$

where $\Phi_{\langle f\rangle}(A) = \{x \in \mathbb{R} : x \text{ is an } f\text{-density point of } A\}$.

It was proved in paper [?] that for any function $f : \mathbb{R}_+ \to \mathbb{R}_+$, the family $\mathcal{T}_{\langle f\rangle}$ is a topology if and only if $\limsup_{x\to 0^+} \frac{x}{f(x)} > 0$. An analogous property occurs for a symmetrical f-density. It also turns out that a symmetrical f-density can not be characterized by one-sided f-density. There exists a set $A \in \mathcal{L}$ and a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\limsup_{x\to 0^+} \frac{x}{f(x)} > 0$ and 0 is a symmetrical f-density point of A and

$$\lim_{h\to 0^+} \frac{\lambda(A' \cap [0, h])}{f(h)} > 0.$$

Since $\Phi_{\langle f\rangle}(A) \subset \Phi^s_{\langle f\rangle}(A)$ for $A \in \mathcal{L}$ we have $\mathcal{T}_{\langle f\rangle} \subset \mathcal{T}^s_{\langle f\rangle}$. The example in paper [?] shows that this inclusion is proper. However, for a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\limsup_{x\to 0^+} \frac{x}{f(x)} > 0$ we get $\mathcal{T}_{\langle f\rangle} = \mathcal{T}^s_{\langle f\rangle}$ if and only if $\Phi_{\langle f\rangle} = \Phi^s_{\langle f\rangle}$.

If

$$\mathcal{A}_1 = \left\{ f : \mathbb{R}_+ \to \mathbb{R}_+ : 0 < \limsup_{x\to 0^+} \frac{x}{f(x)} < \infty \right\},$$

then for every function $f \in \mathcal{A}_1$ the operator $\Phi^s_{\langle f\rangle}$ is the lower density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle$. If $f \in \mathcal{A}_1$ and moreover, $\liminf_{x\to 0^+} \frac{x}{f(x)} > 0$ we get that $\Phi^s_{\langle f\rangle} = \Phi_d$, which implies that $\mathcal{T}^s_{\langle f\rangle} = \mathcal{T}_d$.

It was proved in [?] that the topology $\mathcal{T}^s_{\langle f\rangle}$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\limsup_{x\to 0^+} \frac{x}{f(x)} > 0$, is invariant with respect to the linear operation $\alpha x + b$ for $\alpha \geq 1$ and $b \in \mathbb{R}$. For every $\alpha \in (0, 1)$ there exists a function f with the above properties and a set $A \in \mathcal{T}^s_{\langle f\rangle}$ such that $\alpha A \notin \mathcal{T}^s_{\langle f\rangle}$.

2.6 On the density topologies associated with an extension of the Lebesgue measure

Studies the density topologies related to an extension of the Lebesgue measure have been initialed in my habilitation thesis. I was inspired to study this topic by Professor A. B. Kharazishvili from the University of Tbilisi in Georgia during my stay at this University in 1992. Later, I had been working with Professor A. B. Kharazishvili during his stay at Institute of Mathematics at the University of Łódź in 1995–1997. The results of these studies are presented in papers [?], [?] and [?].

If μ is an extension of the Lebesgue measure on $\mathbb{R}$, then the symbol $\mathcal{S}_\mu$ and $\mathcal{I}_\mu$ will be always mean the σ-algebra of μ-measurable sets and the σ-ideal of μ-null sets, respectively. Thus $\mathcal{S}_\mu = \operatorname{dom}\mu$ and $\mathcal{I}_\mu = \{A \in \mathcal{S}_\mu : \mu(A) = 0\}$.

If μ is an extension of the Lebesgue measure on $\mathbb{R}$, then for every set $A \in \mathcal{S}_\mu$ a point $x_0 \in \mathbb{R}$ is a μ-density point of A, if

$$\lim_{h\to 0} \frac{\mu(A \cap [x_0 - h, x_0 + h])}{2h} = 1.$$

Putting, for any $A \in \mathcal{S}_\mu$

$$\Phi_\mu(A) = \{x \in \mathbb{R} : x \text{ is a } \mu\text{-density point of } A\},$$

we get immediately that Φ_μ is the semi lower density operator on $\langle \mathbb{R}, \mathcal{S}_\mu, \mathcal{I}_\mu \rangle$ and $\Phi_\mu(A) \in F_{\sigma\delta}$ for any $A \in \mathcal{S}_\mu$.

Let

$$\mathcal{T}_\mu = \{A \in \mathcal{S}_\mu : A \subset \Phi_\mu(A)\}.$$

Using the method included in monograph A. B. Kharazishvili *Invariant extension of the Lebesgue measure* (Izd. Tbil. Gos. Univ., Tbilisi, 1983) with application of Vitali covering theorem, I presented in paper [?] the proof that $\mathcal{T}_\mu$ is a topology containing $\mathcal{T}_d$. From this proof it can also be inferred that

$$\mathcal{T}_\mu = \mathcal{T}_d \ominus \mathcal{I}_\mu.$$

The above equality was surprising for me. It means that only sets from the σ-ideal $\mathcal{I}_\mu$ are derived from to the domain of the extension μ to form the topology $\mathcal{T}_\mu$. Moreover, if $A \in \mathcal{S}_\mu$ and $A \subset \Phi_\mu(A)$, then $A \in \mathcal{T}_\mu$. Hence $A = B \setminus C$, where $B \in \mathcal{T}_d$ and $C \in \mathcal{I}_\mu$. It implies that

$$\Phi_\mu(A) \setminus A = \Phi_\mu(B \setminus C) \setminus (B \setminus C) = \Phi_d(B) \setminus (B \setminus C) \in \mathcal{I}_\mu.$$

Therefore Φ_μ is the weak almost lower density operator on $\langle \mathbb{R}, \mathcal{S}_\mu, \mathcal{I}_\mu \rangle$. I observed this property much later when I was considering different type of the density operators. Until now I have not solved, whether Φ_μ is the almost lower density operator on $\langle \mathbb{R}, \mathcal{S}_\mu, \mathcal{I}_\mu \rangle$.

Paper [?] is discussing some properties of the $\mathcal{T}_\mu$ topology. Among the other it is shown that

$$\mathcal{B}a(\mathcal{T}_\mu) = \mathcal{B}(\mathcal{T}_\mu) = \mathcal{L} \triangle \mathcal{I}_\mu.$$

Moreover, I proved that $\mathcal{T}_\mu$ is the von-Neumana topology associated with the measure μ if and only if

$$\mathcal{S}_\mu = \mathcal{L} \triangle \mathcal{I}_\mu.$$

Similarly, Φ_μ is the lower density operator on $\langle \mathbb{R}, \mathcal{S}_\mu, \mathcal{I}_\mu \rangle$ if and only if $\mathcal{S}_\mu = \mathcal{L} \triangle \mathcal{I}_\mu$. In the case of continuous functions we have that

$$\mathcal{C}(\langle \mathbb{R}, \mathcal{T}_\mu \rangle, \langle \mathbb{R}, \tau_{nat} \rangle) = \mathcal{C}(\langle \mathbb{R}, \mathcal{T}_d \rangle, \langle \mathbb{R}, \tau_{nat} \rangle).$$

Paper [?] is raising further properties of the topological space $\langle \mathbb{R}, \mathcal{T}_\mu \rangle$. I showed in it that this space is connected, and it is neither separable nor having the Lindelöf property. Evidently, $\langle \mathbb{R}, \mathcal{T}_\mu \rangle$ is a Hausdorff space for any extension μ, while it is completely regular if and only if $\mathcal{T}_d = \mathcal{T}_\mu$. Assuming C.H., the topological space is not $\langle \mathbb{R}, \mathcal{T}_\mu \rangle$ is not a Blumberg space.

My next research related the $\mathcal{T}_\mu$ topology associated with the Luzin-Menchoff property. Let τ_1 and τ_2 be topological spaces on X and $\tau_1 \subset \tau_2$. We shall say that τ_2 has the Luzin-Menchoff property with respect to τ_1, if for every disjoint sets $F_{\tau_1}, F_{\tau_2} \subset X$ such that F_{τ_1} is τ_1-closed and F_{τ_2} is τ_2-closed there exist $G_{\tau_1}, G_{\tau_2} \subset X$ such that G_{τ_1} is τ_1-open and G_{τ_2} is τ_2-open and $F_{\tau_1} \subset G_{\tau_1}$ and $F_{\tau_2} \subset G_{\tau_2}$. It was proved that $\mathcal{T}_\mu$ does not possess the Luzin-Menchoff property with respect to $\mathcal{T}_d$. Simultaneously, it turned out that $\mathcal{T}_\mu$ possesses the Luzin-Menchoff with respect to τ_{nat} if and only if $\mathcal{T}_\mu = \mathcal{T}_d$.

My habilitation thesis in [?] contains the interesting result there exists the invariant pair $\langle \mathcal{S}, \mathcal{J} \rangle$ such that for every $1 \leq k \leq \omega_0$ there exists a set $A \in \mathcal{S} \setminus \mathcal{J}$ with $\mathrm{card}(\Phi_{\mathcal{J}}(A)) = k$. Such statement is the results of paper [?]. There is constructed an extension μ of the Lebesgue measure such that the pair $\langle \mathcal{S}_\mu, \mathcal{I}_\mu \rangle$ is invariant with respect to the linear operation $qx + b$, where $q \in \mathbb{Q}$, $b \in \mathbb{R}$ and for every $1 \leq k \leq \omega$ there exists a set $A \in \mathcal{S}_\mu \setminus \mathcal{I}_\mu$, such that $\mathrm{card}(\Phi_\mu(A)) = k$. Such result illustrates that Φ_μ is not the lower density operator on $\langle X, \mathcal{S}_\mu, \mathcal{I}_\mu \rangle$, which in effect means that

$$\mathcal{S}_\mu \neq \mathcal{L} \triangle \mathcal{I}_\mu.$$

In paper [?] we consider the semi lower density operator $\Phi^* : \mathcal{S}_\mu \to \mathcal{L}$ on $\langle \mathbb{R}, \mathcal{S}_\mu, \mathcal{I}_\mu \rangle$ having the majorant Φ_μ such that

$$\underset{A \in \mathcal{S}_\mu}{\forall} \; \Phi^*(A) \subset \Phi_\mu(A),$$

where

$$\Phi_\mu(A) = \left\{ x \in \mathbb{R} : \lim_{h \to 0} \frac{\mu(A \cap [x-h, x+h])}{2h} = 1 \right\}$$

for any $A \in \mathcal{S}_\mu$. In fact Φ^* is the weak almost lower density operator, because for any $A \in \mathcal{S}_\mu$, if $A \subset \Phi^*(A)$, then $A \subset \Phi_\mu(A)$ which implies that $\Phi_\mu(A) \setminus A \in \mathcal{I}_\mu$. Hence $\Phi^*(A) \setminus A \in \mathcal{I}_\mu$. So that

$$\mathcal{T}_{\Phi^*} = \{A \in \mathcal{S}_\mu : A \subset \Phi^*(A)\}$$

is a topology of the form

$$\mathcal{T}_{\Phi^*} = \mathcal{T}_{\Phi_\mu} \ominus \mathcal{I}_\mu.$$

In addition the restriction of the operator Φ^* to the σ-algebra $\mathcal{L}$, denoted $\Phi = \Phi^* | \mathcal{L}$, is the almost lower density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L} \rangle$ generating the $\mathcal{T}_\Phi$ topology.

In paper [?] for every function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\limsup_{x \to 0^+} \frac{x}{g(x)} > 0$$

and for any extension μ of the Lebesgue measure it was defined that $x_0 \in \mathbb{R}$ is a g-density point of a set $A \in \mathcal{S}_\mu$, if

$$\lim_{h \to 0^+} \frac{\mu(A' \cap [x_0 - h, x_0 + h])}{g(2h)} = 0.$$

Let

$$\Phi^{\mu}_{\langle g\rangle}(A) = \{x \in \mathbb{R} : x \text{ is a } g\text{-density point of } A\}$$

for $A \in \mathcal{S}_\mu$. The obtained result says that for every function $g : \mathbb{R}_+ \to \mathbb{R}_+$ with the above properties the family

$$\mathcal{T}^{\mu}_{\langle g\rangle} = \{A \in \mathcal{S}_\mu : A \subset \Phi^{\mu}_{\langle g\rangle}(A)\}$$

is a topology on $\mathbb{R}$ such that

$$\mathcal{T}^{\mu}_{\langle g\rangle} = \mathcal{T}_{\langle g\rangle} \ominus \mathcal{I}_\mu,$$

where $\mathcal{T}_{\langle g\rangle} = \{A \in \mathcal{L} : A \subset \Phi_{\langle g\rangle}(A)\}$. However, if we set up with

$$\limsup_{x\to 0^+} \frac{x}{g(x)} = 0$$

then the operator $\Phi^{\mu}_{\langle g\rangle}$ generates the topology $\mathcal{T}^{\mu}_{\langle g\rangle}$ such that

$$\mathcal{T}_{\langle g\rangle} \ominus \mathcal{I}_\mu \subset \mathcal{T}^{\mu}_{\langle g\rangle}.$$

The density type topologies related to an extension of the Lebesgue measure are also the subject of research in paper [?]. The base of considerations in this paper is approaching to a density point by a sequence of closed intervals convergent to zero. It will be discussed in more details in the next chapter. In this case, we are taking into account a sequence $J = \{J_n\}_{n\in\mathbb{N}}$ of closed and non-degenerate intervals such that $0 \in J_n$ for $n \in \mathbb{N}$ and $|J_n| \underset{n\to\infty}{\longrightarrow} 0$. Let $\mathfrak{J}^0$ be the family of such a sequences of closed intervals.

Let $J^0 = \{J^0_n\}_{n\in\mathbb{N}} \in \mathfrak{J}^0$. Let μ be an extension of the Lebesgue measure and $A \in \mathcal{S}_\mu$. We shall say that $x_0 \in \mathbb{R}$ is a J^0-density point of A, if

$$\lim_{n\to\infty} \frac{\mu(A \cap (x_0 + J_n))}{|J_n|} = 1.$$

Putting, for $A \in \mathcal{S}_\mu$

$$\Phi^{\mu}_{J^0}(A) = \{x \in \mathbb{R} : x \text{ is a } J^0\text{-density point of } A\},$$

we get that $\Phi^{\mu}_{J^0} : \mathcal{S}_\mu \to \mathcal{L}$ is the weak almost lower density operator on $\langle \mathbb{R}, \mathcal{S}_\mu, \mathcal{I}_\mu \rangle$. This makes it possible to deduce that

$$\mathcal{T}^{\mu}_{J^0} = \{A \in \mathcal{S}_\mu : A \subset \Phi^{\mu}_{J^0}(A)\}$$

is a topology containing $\mathcal{T}_d$. If $\mu = \lambda$ we get the topology

$$\mathcal{T}_{J^0} = \{A \in \mathcal{L} : A \subset \Phi_{J^0}(A)\}$$

and there is the property that

$$\mathcal{T}^{\mu}_{J^0} = \mathcal{T}_{J^0} \ominus \mathcal{I}_\mu.$$

For every sequence$J \in \mathfrak{J}^0$ we also receive that Φ^{μ}_{J} is the lower density operator on $\langle \mathbb{R}, \mathcal{S}_\mu, \mathcal{I}_\mu \rangle$ if and only if $\mathcal{S}_\mu = \mathcal{L} \triangle \mathcal{I}_\mu$, which one is equivalent to the property that $\mathcal{T}^{\mu}_{J}$ is the von-Neumanna topology associated with μ.

For any sequence $J \in \mathfrak{J}^0$ the topological space $\langle \mathbb{R}, \mathcal{T}^{\mu}_{J} \rangle$ is a Hausdorff space and it is not a normal space. The separation axioms: regularity and complete regularity referring to the topological space $\langle \mathbb{R}, \mathcal{T}^{\mu}_{J} \rangle$ is satisfied if and only if $\mathcal{I}_\mu = \mathbb{L}$. Here it is worth mentioning the result of A. B. Kharazishvili in the paper *A non separable extension of the Lebesgue measure without new null sets* (Real Anal. Exchange 33, no 1, 259–268, 2008) saying that under C. H. there exists a non separable extension μ of the Lebesgue measure on $\mathbb{R}$ such that $\mathcal{I}_\mu = \mathbb{L}$.

Paper [?] contains a general overview on topologies generated by the weak lower density operator connected with an extension μ of the Lebesgue measure. Let $\Phi_\mu : \mathcal{S}_\mu \to 2^{\mathbb{R}}$ be the weak almost lower density operator on $\langle \mathbb{R}, \mathcal{S}_\mu, \mathcal{I}_\mu \rangle$. The operator Φ_μ generates the topology

$$\mathcal{T}_{\Phi_\mu} = \{A \in \mathcal{S}_\mu : A \subset \Phi_\mu(A)\}.$$

If $\Phi = \Phi_\mu | \mathcal{L}$ and Φ is the lower density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L} \rangle$, then

$$\mathcal{T}_{\Phi_\mu} = \mathcal{T}_{\Phi} \ominus \mathcal{I}_\mu,$$

where $\mathcal{T}_\Phi = \{A \in \mathcal{L} : A \subset \Phi(A)\}$ is a topology generated by Φ. We infer the property that for every weak almost lower density operator Φ_μ on $\langle \mathbb{R}, \mathcal{S}_\mu, \mathcal{I}_\mu \rangle$, if $\Phi = \Phi_\mu | \mathcal{L}$

is the lower density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L} \rangle$, then the topology $\mathcal{T}_\mu$ generated by Φ_μ is an abstract density topology on $\langle \mathbb{R}, \mathcal{L} \triangle \mathcal{I}_\mu, \mathcal{I}_\mu \rangle$. Moreover, $\mathcal{T}_\mu$ is the von-Neumanna topology associated with μ if and only if $\mathcal{S}_\mu = \mathcal{L} \triangle \mathcal{I}_\mu$.

We shall say that the topological spaces $\langle X, \tau_1 \rangle$ and $\langle X, \tau_2 \rangle$ are similar, which was defined in paper of A. Bartoszewicz, M. Filipczak, A. Kowalski, M. Terepeta, *On similarity between topologies* (Cent. Eur. J. Math. 12 (2014), no. 4, 603–610), if

$$\underset{A \subset X}{\forall} (\mathrm{int}_{\tau_1}(A) \neq \emptyset \Leftrightarrow \mathrm{int}_{\tau_2}(A) \neq \emptyset).$$

Let μ_1 and μ_2 be the extensions of the Lebesgue measure and Φ_{μ_1}, Φ_{μ_2} are the almost lower density operators on $\langle X, \mathcal{S}_{\mu_1}, \mathcal{I}_{\mu_1} \rangle$ and $\langle X, \mathcal{S}_{\mu_2}, \mathcal{I}_{\mu_2} \rangle$ generating topologies $\mathcal{T}_{\Phi_{\mu_1}}$ and $\mathcal{T}_{\Phi_{\mu_2}}$, respectively, and $\Phi_{\mu_1}|\mathcal{L}$, $\Phi_{\mu_2}|\mathcal{L}$ are the lower density operators on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L} \rangle$. Then the topological spaces $\langle \mathbb{R}, \mathcal{T}_{\Phi_{\mu_1}} \rangle$ and $\langle \mathbb{R}, \mathcal{T}_{\Phi_{\mu_2}} \rangle$ are similar if and only if $\mathcal{T}_{\Phi_{\mu_1}} = \mathcal{T}_{\Phi_{\mu_2}}$.

Let $\mathcal{T}_{\Phi_\mu}$ be the topology generated by the weak lower density operator Φ_μ on $\langle \mathbb{R}, \mathcal{S}_\mu, \mathcal{I}_\mu \rangle$ and $\Phi = \Phi_\mu|\mathcal{L}$ be the lower density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L} \rangle$. In the area of the separation axioms we have:

(i) $\langle \mathbb{R}, \mathcal{T}_{\Phi_\mu} \rangle$ is T_1-space.

(ii) $\langle \mathbb{R}, \mathcal{T}_{\Phi_\mu} \rangle$ is T_2-space if and only if $\langle \mathbb{R}, \mathcal{T}_\Phi \rangle$ is T_2-space.

(iii) If $\langle \mathbb{R}, \mathcal{T}_\Phi \rangle$ is $T_3(T_{3\frac{1}{2}})$-space, then $\langle \mathbb{R}, \mathcal{T}_{\Phi_\mu} \rangle$ is $T_3(T_{3\frac{1}{2}})$-space if and only if $\mathcal{T}_{\Phi_\mu} = \mathcal{T}_\Phi$.

(iv) If $\tau_{nat} \subset \mathcal{T}_\Phi$, then $\langle \mathbb{R}, \mathcal{T}_{\Phi_\mu} \rangle$ is not T_4-space.

Paper [**?**] is related to an extension of the given measure, but it does not concern the topologies generated by such an extension. Let $\langle X, \mathcal{S}, \nu \rangle$ be a measurable space with a σ-finite measure ν, with the notation that $\mathcal{S} = \operatorname{dom} \nu$. Then for every set $Y \subset X$ and arbitrary $c \in [\nu_*(Y), \nu^*(Y)]$, whereas ν_* and ν^* denote the inner and outer measure generated by ν, respectively, there exists an extension ν_c of the measure ν such that Y is a member of the smallest σ-algebra generated by $\mathcal{S} \cup \{Y\}$ and $\nu_c(Y) = 0$. For every $Y \subset X$ the cardinality of such extensions is equal to 1 or is not less than continuum cardinality.

2.7 On the density topologies generated by the sequences of closed intervals convergent to zero

During the presentation of the $\langle s \rangle$-density topologies at University of Joannina in Greece, with Erasmus Program, in 2012, I was asked by Prof. G. Karakostas to consider density not with respect to the sequences of closed symmetrical intervals but according to the sequences of arbitrary measurable sets of the positive measure. A partial answer to this question is presented in paper [?].

We shall say that a sequence $J = \{J_n\}_{n \in \mathbb{N}}$ of closed non-degenerate intervals is convergent to 0, if $\operatorname{diam}(\{0\} \cup J_n) \underset{n\to\infty}{\longrightarrow} 0$, where $\operatorname{diam}(\{0\} \cup J_n)$ is the diameter of the set $\{0\} \cup J_n$. Let $J = \{J_n\}_{n \in \mathbb{N}}$ be a sequence of closed non-degenerate intervals convergent to zero. We shall say that $x_0 \in \mathbb{R}$ is a J-density point of a set $A \in \mathcal{L}$, if

$$\lim_{n\to\infty} \frac{\lambda(A \cap (J_n + x_0))}{|J_n|} = 1.$$

Let

$$\Phi_J(A) = \{x \in \mathbb{R} : x \text{ is a } J\text{-density point of } A\}$$

for $A \in \mathcal{L}$. Then $\Phi_J(A) \in \mathcal{F}_{\sigma\delta}$ for every $A \in \mathcal{L}$.

This type of density one can find in paper of M. Csörnyei, *Density theorems revisited* (Acta Sci. Math. (Szeged) 64 (1998), no. 1-2, 59–65), where it is proved that the analogon of the Lebesgue theorem is not true for the density J-density points. So that there exists a sequence J of closed intervals tending to zero such that Φ_J is not the lower density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L} \rangle$.

Let $\mathfrak{J}$ be the family of all sequences of closed, non-degenerate intervals convergent to zero. Paper [?] is showing that for an arbitrary $J \in \mathfrak{J}$ the operator $\Phi_J : \mathcal{L} \to \mathcal{L}$ is the almost lower density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L} \rangle$ and it generates the topology

$$\mathcal{T}_J = \{A \in \mathcal{L} : A \subset \Phi_J(A)\}.$$

The topology $\mathcal{T}_J$ is called the J-density topology and it is stronger than τ_{nat}. These J-density topologies are the main feature of investigation in further papers.

To distinguish certain operators, the following expression was defined:

$$\alpha(J) = \limsup_{n\to\infty} \frac{\operatorname{diam}(\{0\} \cup J_n)}{|J_n|}.$$

It turned out that the condition $\alpha(J) < \infty$ is equivalent to the condition that $\mathcal{T}_d \subset \mathcal{T}_J$. At the same time the result in paper [?] shows that for any sequence $J \in \mathfrak{J}$ such that $\alpha(J) = \infty$ there exists a set $A \in \tau_{nat}$ such that $0 \in \Phi_d(A)$ and $0 \notin \Phi_J(A)$, and thus $A \cup \{0\} \in \mathcal{T}_d$ and $A \cup \{0\} \notin \mathcal{T}_J$.

The most important result of paper [?] is the theorem that for each sequence $J \in \mathfrak{J}$ with $\alpha(J) < \infty$ the operator $\Phi_J : \mathcal{L} \to \mathcal{L}$ is the lower density operator on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L} \rangle$. The property illustrating that there exist a sequence $J \in \mathfrak{J}$ and the sets $A, B \in \mathcal{L}$ such that $0 \in \Phi_J(A)$, $0 \notin \Phi_d(A)$ and $0 \in \Phi_d(B)$, $0 \notin \Phi_J(B)$ justifies that we received a new concept of density points. Clearly, if $J = \{J_n\}_{n\in\mathbb{N}}$, with $J_n = [-\frac{1}{s_n}, \frac{1}{s_n}]$ for $n \in \mathbb{N}$ and $\langle s \rangle = \{s_n\}_{n\in\mathbb{N}} \in \mathbb{S}$ we get that a J-density is a generalization of an $\langle s \rangle$-density.

In paper [?], the results of research on real $\mathcal{T}_J$-continuous functions were presented. For every sequence $J \in \mathfrak{J}$ the family of all $\mathcal{T}_J$-continuous functions is closed under the basic algebraic operations. Analogously as in the case of approximate continuity the concept of J-approximate continuity was defined. We shall say that a function $f : \mathbb{R} \to \mathbb{R}$ is J-approximately continuous at $x_0 \in \mathbb{R}$, if there exists a set $A \in \mathcal{L}$ such that $x_0 \in \Phi_J(A)$ and $f_{|(A\cup\{x_0\})}$ is continuous at x_0 with respect to τ_{nat}. It was proved that for $\alpha(J) < \infty$ a function $f : \mathbb{R} \to \mathbb{R}$ is J-approximately continuous everywhere if and only if f is $\mathcal{T}_J$-continuous. Paper [?] also contains the result related to the conditions equivalent to the measurability of a function in the Lebesgue sense. Namely, for a function $f : \mathbb{R} \to \mathbb{R}$ the following conditions are equivalent:

a) f is measurable in the Lebesgue sense,

b) there exists $B \in \mathbb{L}$ such that for every sequence $J \in \mathfrak{J}$ with $\alpha(J) < \infty$ and any $x \in \mathbb{R} \setminus B$ the function f is J-approximately continuous at x,

c) there exist a sequence $J \in \mathfrak{J}$ with $\alpha(J) < \infty$ and $B_J \in \mathbb{L}$ such that f is J-approximately continuous for $x \in \mathbb{R} \setminus B_J$.

In the area of separation axioms paper [**?**] contains the result that for a sequence $J \in \mathfrak{J}$ such that $\alpha(J) < \infty$, the topological space $\langle \mathbb{R}, \mathcal{T}_J \rangle$ is regular and it is not normal. In the case of the Baire classification, there is the result that for any $J \in \mathfrak{J}$ such that $\alpha(J) < \infty$ every J-approximately continuous function $f : \mathbb{R} \to \mathbb{R}$ belongs to the 1 Baire class.

Paper [**?**] contains further results of research on $\mathcal{T}_J$-continuity. Let $J \in \mathfrak{J}$. We shall say that a sequence $J = \{[a_n, b_n]\}_{n \in \mathbb{N}}$ is right-side (left-side) convergent to zero, if there exists $n_0 \in \mathbb{N}$ such that $b_n > 0$ $(a_n < 0)$ for $n \geq n_0$ and

$$\lim_{n\to\infty} \frac{\min\{0, a_n\}}{b_n} = 0 \; \left(\lim_{n\to\infty} \frac{\max\{0, b_n\}}{a_n} = 0 \right).$$

In the context of one-side convergent sequences we get that if $J \in \mathfrak{J}$, then $[a, b) \in \mathcal{T}_J$ $((a, b] \in \mathcal{T}_J)$ for $a, b \in \mathbb{R}, a < b$ if and only if J is right-side (left-side) convergent to 0.

For $J \in \mathfrak{J}$ we define the following families of continuous functions:

$$\begin{aligned} \mathcal{C}_{nat,nat} &= \{f : \langle \mathbb{R}, \tau_{nat} \rangle \to \langle \mathbb{R}, \tau_{nat} \rangle\}, \\ \mathcal{C}_{nat,J} &= \{f : \langle \mathbb{R}, \tau_{nat} \rangle \to \langle \mathbb{R}, \mathcal{T}_J \rangle\}, \\ \mathcal{C}_{J,nat} &= \{f : \langle \mathbb{R}, \mathcal{T}_J \rangle \to \langle \mathbb{R}, \tau_{nat} \rangle\}, \\ \mathcal{C}_{J,J} &= \{f : \langle \mathbb{R}, \mathcal{T}_J \rangle \to \langle \mathbb{R}, \mathcal{T}_J \rangle\}. \end{aligned}$$

The family $\mathcal{C}_{nat,J}$ reduces to the constant functions. At the same time $\mathcal{C}_{nat,J} \subsetneq \mathcal{C}_{nat,nat} \subset \mathcal{C}_{J,nat}$ and $\mathcal{C}_{nat,J} \subsetneq \mathcal{C}_{J,J} \subset \mathcal{C}_{J,nat}$. If a sequence $J \in \mathfrak{J}$ is one-side convergent to zero, then

i) $\mathcal{C}_{nat,nat} \setminus \mathcal{C}_{J,J} \neq \emptyset$,

ii) $\mathcal{C}_{J,J} \setminus \mathcal{C}_{nat,nat} \neq \emptyset$

and inclusions $\mathcal{C}_{nat,nat} \subset \mathcal{C}_{J,nat}$ and $\mathcal{C}_{J,J} \subset \mathcal{C}_{J,nat}$ are proper.

Another interesting thing is the result important in semiregularization of a given topology. For any sequence $J \in \mathfrak{J}$ there exists an interval set $B = \bigcup_{n=1}^{\infty} (a_n, b_n)$, where $b_{n+1} < a_n < b_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = 0$, such that $0 \in \Phi_J(B)$. However, in the direction of comparison of J-density topologies the result in paper [?] says that for any sequence $J \in \mathfrak{J}$ there exists a sequence $K \in \mathfrak{J}$ such that topologies $\mathcal{T}_J$ and $\mathcal{T}_K$ are not comparable.

For the sequences $J, K \in \mathfrak{J}$ a sequence obtained from all intervals of J and K in any way will be denoted by $J \cup K$. We have always that $J \cup K \in \mathfrak{J}$. In the context of the sequence $J \cup K$ we get the following properties:

i) $\mathcal{C}_{J,nat} \cap \mathcal{C}_{K,nat} = \mathcal{C}_{J \cup K,nat}$,

ii) $\mathcal{C}_{nat,J} \cap \mathcal{C}_{nat,K} = \mathcal{C}_{nat,J \cup K}$,

iii) $\mathcal{C}_{J,J} \cap \mathcal{C}_{K,K} \subsetneq \mathcal{C}_{J \cup K,J \cup K}$.

Paper [?] contains further continuation of studies on J-density topologies. New results in this paper concern the study of J-approximately continuous functions for any sequence $J \in \mathfrak{J}$ without any restrictions on $\alpha(J)$. A J-approximately continuous function we define analogically as in [?] and we get the result that a function $f : \mathbb{R} \to \mathbb{R}$ is J-approximately continuous if and only if f is $\mathcal{T}_J$-continuous. In this paper there is also the characterization of measurability in the Lebesgue sense. A function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable if and only if there exists a sequence $J \in \mathfrak{J}$ and a set $B_J \in \mathbb{L}$ such that f is J-approximately continuous in each point $x \in \mathbb{R} \setminus B_J$. Also, for every sequence $J \in \mathfrak{J}$ a J-approximately continuous function $f : \mathbb{R} \to \mathbb{R}$ is the first Baire class. The result in the area of separation axioms shows that for every $J \in \mathfrak{J}$ with $\alpha(J) < \infty$ the topological space $\langle \mathbb{R}, \mathcal{T}_J \rangle$ is completely regular. Under the condition $\alpha(J) < \infty$ we have $\mathcal{T}_d \subset \mathcal{T}_J$, which is presented in paper [?]. Because there is a sequence $J \in \mathfrak{J}$ such that $\alpha(J) < \infty$ and $\mathcal{T}_d \neq \mathcal{T}_J$, we have the example of J-density topology, different from the density topology and completely regular.

The concept J-density with respect to sequences of closed non-degenerate intervals convergent to zero have the further generalization. In the paper of R. Wiertelak, F. Strobin *On a generalization of density topologies on the real line* (Topology Appl. 199, 1–16, 2016) is presented the concept of a density with respect to the sequences of positive Lebesgue measurable sets convergent to zero. It turned out, that the density with respect to such a sequence can be described by the density with respect to a sequence of sets convergent to zero whose terms are the finite unions of closed intervals. Research over the topologies generated by such sequences are the main interest of PhD student M. Widzibor who is preparing doctoral desertion under my supervision.

Before the study of the J-density in the aspect of measure, in the doctoral dissertation of R. Wiertelak *Density topologies by sequences of intervals convergent to zero* prepared under my supervision in 2008, the category aspect of the density related to the sequences of closed non-degenerated intervals convergent to zero was introduced.

If $J = \{J_n\}_{n\in\mathbb{N}} \in \mathfrak{J}$, then we shall say that $x_0 \in \mathbb{R}$ is an $\mathcal{I}(J)$-density point of a set $A \in \mathcal{B}a$ if the sequence of functions $\{h(A, J_n)\}_{n\in\mathbb{N}}$ is convergent with respect to σ-ideal $\mathbb{K}$ to the function $\chi_{[-1,1]}$, where for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$h(A, J_n)(x) = \chi_{\frac{2}{|J_n|}((A-x_0)-s(J_n))\cap[-1,1]}(x)$$

and $s(J_n)$ is the center of the interval J_n.

In the case of $J = \{[-\frac{1}{n}, \frac{1}{n}]\}_{n\in\mathbb{N}}$ we get, of course, that the $\mathcal{I}(J)$-density coincides with the $\mathcal{I}$-density. For $J = \{[-\frac{1}{s_n}, \frac{1}{s_n}]\}_{n\in\mathbb{N}}$, where $\langle s\rangle = \{s_n\}_{n\in\mathbb{N}} \in \mathbb{S}$, we get the $\mathcal{I}_{\langle s\rangle}$-density.

Putting, for every $J \in \mathfrak{J}$ and every $A \in \mathcal{B}a$

$$\Phi_{\mathcal{I}(J)}(A) = \{x \in \mathbb{R} : x \text{ is an } \mathcal{I}(J)\text{-density point of } A\}$$

R. Wiertelak proved that $\Phi_{\mathcal{I}(J)} : \mathcal{B}a \to \mathcal{B}a$ is the lower density operator on $\langle\mathbb{R}, \mathcal{B}a, \mathbb{K}\rangle$ for any $J \in \mathfrak{J}$. It implies that for any $J \in \mathfrak{J}$

$$T_{\mathcal{I}(J)} = \{A \in \mathcal{B}a : A \subset \Phi_{\mathcal{I}(J)}(A)\}$$

is a topology stronger than τ_{nat} and it is called the $\mathcal{T}_{\mathcal{I}(J)}$-density topology.

Paper [?] contains the results related to $\mathcal{T}_{\mathcal{I}(J)}$-continuous functions. There were investigated the connection between the following classes of functions:

$$\mathcal{C}_{nat,\mathcal{I}(J)} = \{f : \langle\mathbb{R}, \tau_{nat}\rangle \to \langle\mathbb{R}, \mathcal{T}_{\mathcal{I}(J)}\rangle\},$$
$$\mathcal{C}_{\mathcal{I}(J),nat} = \{f : \langle\mathbb{R}, \mathcal{T}_{\mathcal{I}(J)}\rangle \to \langle\mathbb{R}, \tau_{nat}\rangle\},$$
$$\mathcal{C}_{\mathcal{I}(J),\mathcal{I}(J)} = \{f : \langle\mathbb{R}, \mathcal{T}_{\mathcal{I}(J)}\rangle \to \langle\mathbb{R}, \mathcal{T}_{\mathcal{I}(J)}\rangle\},$$
$$\mathcal{C}_{nat,nat} = \{f : \langle\mathbb{R}, \tau_{nat}\rangle \to \langle\mathbb{R}, \tau_{nat}\rangle\}.$$

It is interesting to note, that a linear function $f(x) = ax$ is $\mathcal{T}_{\mathcal{I}(J)}$-continuous for every $J \in \mathfrak{J}$ if and only if $a \in \{0, 1\}$. Another characterization of continuity of a linear function says that $f(x) = ax$ for $a \neq 0$ is $\mathcal{T}_{\mathcal{I}(J)}$-continuous for every $J \in \mathfrak{J}$ if and only if $\mathcal{T}_{\mathcal{I}(J)} \subset \mathcal{T}_{\mathcal{I}(aJ)}$, where $aJ = \{aJ_n\}_{n\in\mathbb{N}}$.

Considerations of J-density operators, for the measure and the category, increase collections of differences between the measure and the category. For $J \in \mathfrak{J}$ an operator $\Phi_J : \mathcal{L} \to \mathcal{L}$ is the almost lower density operator on $\langle\mathbb{R}, \mathcal{L}, \mathbb{L}\rangle$, while $\Phi_{\mathcal{I}(J)} : \mathcal{B}a \to \mathcal{B}a$ is the lower density operator on $\langle\mathbb{R}, \mathcal{B}a, \mathbb{K}\rangle$. In both cases, the operators generate $\mathcal{T}_J$-density and $\mathcal{T}_{\mathcal{I}(J)}$-density topology, respectively. The first of which, when $\alpha(J) < \infty$, is completely regular and other one is not regular.

2.8 Semiregularization

Papers [?] and [?] are related to the semiregularization of the density type topologies. If $\langle X, \tau\rangle$ is a topological space, then the semiregularization of $\langle X, \tau\rangle$ is the topological space $\langle X, \tau_{sem}\rangle$, where τ_{sem} is the topology generated by τ-regular open sets. An important feature of the semiregularization is the theorem that, that if $\langle X, \tau_{sem}\rangle$ is the semiregularization of $\langle X, \tau\rangle$ and $\langle Y, \sigma\rangle$ is a regular topological space, then there is the equality of classes of continuous functions:

$$\mathcal{C}(\langle X, \tau\rangle, \langle Y, \sigma\rangle) = \mathcal{C}(\langle X, \tau_{sem}\rangle, \langle Y, \sigma\rangle),$$

(R. A. Aleksandrian, E. M. Mirzahanian, *General topology* (in Russian), Moscow, 1979). In connection with the above theorem, it turns out that,if the space $\langle X, \tau_{sem}\rangle$ is completely regular, then the semiregularization is the coarsest topology with the property that

$$\mathcal{C}(\langle X, \tau\rangle, \langle Y, \sigma\rangle) = \mathcal{C}(\langle X, \tau_{sem}\rangle, \langle Y, \sigma\rangle).$$

It seams important to look for the coarsest topology τ for which

$$\mathcal{C}(\langle \mathbb{R}, \tau\rangle, \langle \mathbb{R}, \tau_{nat}\rangle) = \mathcal{C}(\langle \mathbb{R}, \mathcal{T}_\Phi\rangle, \langle \mathbb{R}, \tau_{nat}\rangle),$$

where $\mathcal{T}_\Phi$ is a topology generated by different types of operators Φ. It is known, from paper [?], that a topological space $\langle \mathbb{R}, \mathcal{T}_\Phi\rangle$, where $\mathcal{T}_\Phi$ is a topology generated by the lower density operator on $\langle \mathbb{R}, \mathcal{B}a, \mathbb{K}\rangle$, with the assumption $\tau_{nat} \subset \mathcal{T}_\Phi$, is not regular. Therefore, in this case it is reasonable to investigate the semiregularization of the topological space $\langle \mathbb{R}, \mathcal{T}_\Phi\rangle$.

Paper [?] contains some overview of the density type topologies generated by the lower density operators on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle$ or $\langle \mathbb{R}, \mathcal{B}a, \mathbb{K}\rangle$, respectively, for which the semiregularization is the completely regular space. This paper includes, among other, the study of the semiregularization of $\mathcal{T}_{\psi_C}$. This topology is associated with an operator depending on a continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{t \to 0^+} \psi(t) = 0.$$

Namely, $x_0 \in \mathbb{R}$ is a ψ_C-density point of a set $E \in \mathcal{B}a$, if

$$\lim_{h \to 0^+} \frac{\lambda((G(E))' \cap [x_0 - h, x_0 + h])}{2h\psi(2h)} = 0,$$

where $G(E)$ is a τ_{nat}-regular open set such that $G(E) \triangle E \in \mathbb{K}$. Putting, for $A \in \mathcal{B}a$

$$\Phi_{\psi_C}(A) = \{x \in \mathbb{R} : x \text{ is a } \psi_C\text{-density point of } A\}$$

we get the lower density operator on $\langle \mathbb{R}, \mathcal{B}a, \mathbb{K}\rangle$. It implies that Φ_{ψ_C} generates the topology $\mathcal{T}_{\psi_C}$. Paper [?] contains the result that the semiregularization of the topological space $\langle \mathbb{R}, \mathcal{T}_{\psi_C}\rangle$ is completely regular.

In paper [?] the semiregularization of the density type topology is examined. Let $J \in \mathfrak{J}$. Putting

$$\Phi_r(A) = \Phi_J(G(A)),$$

where $A \in \mathcal{B}a$ and $G(A)$ is a τ_{nat}-regular open set such that $A \triangle G(A) \in \mathbb{K}$, we get the lower density operator on $\langle \mathbb{R}, \mathcal{B}a, \mathbb{K} \rangle$. It implies that Φ_r generates the topology $\mathcal{T}_{\Phi_r}$ stronger than τ_{nat}. The semiregularization of $\langle \mathbb{R}, \mathcal{T}_{\Phi_r} \rangle$ leads to a completely regular space.

Despite some examples of the semiregularization of topological spaces generated by the lower density operators, the problem of the determining condition is not solved for the topology $\mathcal{T}_\Phi$ generated by the lower density operator Φ on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ so that the semiregularization of $\langle X, \mathcal{T}_\Phi \rangle$ was a completely regular space.

2.9 On the families of lower density, almost lower density and semi lower density operators

The paper of A. Bartoszewicz, K. Ciesielski, *MB-representations and topological algebras* (Real Anal. Exchange 27, 749–756, 2001/2002) contains the result of the existence of a measurable space $\langle X, \mathcal{S}, \mathcal{J} \rangle$ with the distinguished σ-ideal $\mathcal{J}$ such that the pair $\langle \mathcal{S}, \mathcal{J} \rangle$ possesses the hull property and there is no the lower density operator on $\langle X, \mathcal{S}, \mathcal{J} \rangle$. It implies that any topology τ on X is no an abstract density topology on $\langle X, \mathcal{S}, \mathcal{J} \rangle$.

In a situation, when we are weakening the task of the existence of the almost lower density operator on $\langle X, \mathcal{S}, \mathcal{J} \rangle$, we can always state that such an operator exists. It shows the simple example:

$$\Phi(A) = \begin{cases} X & A \sim X, \\ \emptyset & \neg(A \sim X) \end{cases}$$

for $A \subset X$, which obviously is the almost lower density operator on $\langle X, \mathcal{S}, \mathcal{J} \rangle$. Moreover, Φ generates the topology

$$\mathcal{T}_\Phi = \{A \subset X : A = \emptyset \vee (A = X \setminus B \wedge B \in \mathcal{J})\}$$

regardless of whether the pair $\langle \mathcal{S}, \mathcal{J} \rangle$ has the hull property.

However, paper [?] contains the previously mentioned result that for any topological space $\langle X, \tau \rangle$ such that $X \notin \mathbb{K}(\tau)$ there is always the lower density operator on

$\langle X, \mathcal{B}a(\tau), \mathbb{K}(\tau)\rangle$. Also,in the case of the space $\langle X, \mathcal{S}_\nu, \mathcal{J}_\nu\rangle$, where $\mathcal{S}_\nu$ is the domain of σ-finite and not identity equal to zero measure ν and $\mathcal{J}_\nu$ is a σ-ideal of ν-zero sets due to the von Neumann-Maharam theorem there is the lower density operator on $\langle X, \mathcal{S}_\mu, \mathcal{J}_\mu\rangle$.

Paper [?] contains the results related to the study of the density type operators on a measurable space with a distinguished algebra and a proper ideal. More precisely, in this paper there is considered a triple $\langle X, \mathcal{A}, \mathcal{I}\rangle$ such that $\mathcal{A}$ is an algebra, $\mathcal{I} \subset \mathcal{A}$ is a proper ideal such that $\bigcup \mathcal{I} = X$. Then, if $\mathcal{P}$ is the family of the lower density operators on $\langle X, \mathcal{A}, \mathcal{I}\rangle$, then the following conditions are equivalent:

a) $\mathcal{A} = \mathcal{A}_0$, where $\mathcal{A}_0$ is an algebra generated by the family $\mathcal{I}$,

b) there exists the smallest, in the sense of inclusion, an abstract density topology on $\langle X, \mathcal{A}, \mathcal{I}\rangle$,

c) $\mathrm{card}(\mathcal{P}) = 1$,

d) the operator $\Phi_{**} : \mathcal{A} \to 2^X$ defined by the formula

$$\mathbf{\Phi}_{**}(A) = \bigcap_{\Phi \in \mathcal{P}} \Phi(A)$$

is the lower density topology on $\langle X, \mathcal{A}, \mathcal{I}\rangle$.

In paper [?] there are investigated the families of all lower density operators, almost lower density operators and semi lower density operators on $\langle X, \mathcal{A}, \mathcal{I}\rangle$.

If $\Phi_1 : \mathcal{A} \to 2^X$ and $\Phi_2 : \mathcal{A} \to 2^X$ are any operators, we are defining the natural order relation $\prec$ saying that $\Phi_1 \prec \Phi_2$, if for any $A \in \mathcal{A}$ we have that $\Phi_1(A) \subset \Phi_2(A)$. If Φ_1, Φ_2 are the lower density operators on $\langle X, \mathcal{A}, \mathcal{I}\rangle$ and $\mathcal{T}_{\Phi_1}$, $\mathcal{T}_{\Phi_2}$ are topologies generated by Φ_1, Φ_2, respectively, then $\mathcal{T}_{\Phi_1} \subset \mathcal{T}_{\Phi_2}$ if and only if $\Phi_1 \prec \Phi_2$.

If $\mathcal{LDO}$ is the family of all lower density operators on $\langle X, \mathcal{A}, \mathcal{I}\rangle$, then assuming that $\mathcal{LDO} \neq \emptyset$, the existence of the largest element in $\mathcal{LDO}$ with respect to the relation $\prec$ is equivalent to the condition that $\mathcal{A} = \mathcal{A}_0$, where $\mathcal{A}_0$ is the algebra generated by $\mathcal{I}$. However, $\Phi \in \mathcal{LDO}$ is a maximal element in $\mathcal{LDO}$ with respect to $\prec$ if and only if Φ

is a lifting on $\langle X, \mathcal{A}, \mathcal{I} \rangle$, that is, the lower density operator on $\langle X, \mathcal{A}, \mathcal{I} \rangle$ satisfying the addition condition that $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$ for any $A, B \in \mathcal{A}$.

In the case of the family of all almost lower density operators on $\langle X, \mathcal{A}, \mathcal{I} \rangle$, denoted by $\mathcal{ALDO}$, we get that if $\Phi \in \mathcal{ALDO}$ is a maximal element in $\mathcal{ALDO}$ with respect to $\prec$, then $\Phi \in \mathcal{LDO}$. Moreover, the following conditions are equivalent:

i) there exists a maximal element in $\mathcal{ALDO}$ with respect to the relation $\prec$,

ii) $\mathcal{LDO} \neq \emptyset$,

iii) there exists an operator $\Phi \in \mathcal{LDO}$ being a lifting.

On the other hand, the existence of the greatest element in $\mathcal{ALDO}$ with respect to the relation $\prec$ is equivalent to the condition that $\mathcal{A} = \mathcal{A}_0$.

In the case of the family of all the semi lower density operators on $\langle X, \mathcal{A}, \mathcal{I} \rangle$, denoted by $\mathcal{SLDO}$, the fact that Φ is a maximal element in $\mathcal{SLDO}$ with respect to the relation $\prec$ is equivalent to the condition that $\Phi(A) \cup \Phi(X \setminus A) = X$ for any $A \in \mathcal{A}$. It should be added that in the family $\mathcal{SLDO}$ always exists a maximal element with respect to $\prec$. However, the existence of the largest element in $\mathcal{SLDO}$ with respect to $\prec$ is equivalent to the condition that $\mathcal{A} = \mathcal{A}_0$.

Paper [?] contains also the result that if the family of topologies generated by the lower density operators on $\langle X, \mathcal{A}, \mathcal{I} \rangle$ is non-empty, then the condition $\mathcal{A} \neq \mathcal{A}_0$ is equivalent to the property that the smallest topology generated by the union of such topologies is 2^X. Similarly, in the case of the family of topologies generated by the almost lower density operators on $\langle X, \mathcal{A}, \mathcal{I} \rangle$ that is the existence of the largest topology is equivalent to the condition $\mathcal{A} = \mathcal{A}_0$.

3 Discussion on the chapters in the monographs

Paper [?] is a chapter in the jubilee monograph *Real functions, density topology and selected topics* (Łódź University Press, 77–82, 2011) dedicated to Professor W. Wilczyński.

There is considered the family $\mathfrak{A}$ of invariant an proper σ-ideals on $\mathbb{R}$. We shall say that a σ-ideal $\mathcal{J} \in \mathfrak{A}$ is universal, if for an arbitrary σ-ideal $\mathcal{I} \in \mathfrak{A}$ we get one of the conditions:

1° $\mathcal{I} \cap \mathcal{B} \subset \mathcal{J} \cap \mathcal{B}$

2° $\mathcal{I}$ and $\mathcal{J}$ are orthogonal, that is, there are $X \in \mathcal{I}$, $Y \in \mathcal{J}$ such that $X \cap Y = \emptyset$ and $X \cup Y = \mathbb{R}$.

This paper contains the result that σ-ideals $\mathbb{L}$ and $\mathbb{K}$ are universal and Borel. It is shown that if σ-ideals $\mathcal{J}_1$, $\mathcal{J}_2$ are universal and Borel, and $\mathcal{J}_1 \neq \mathcal{J}_2$ then $\mathcal{J}_1 \cap \mathcal{J}_2$ in not universal. Moreover, every σ-ideal $\mathcal{J} \in \mathfrak{A}$, being totally imperfect, which means that it does not contain nonempty perfect sets, is not universal. If $\mathcal{J} \in \mathfrak{A}$ is Borel and there exists Borel σ-ideal $\mathcal{J}_1 \in \mathfrak{A}$ such that $\mathcal{J} \subset \mathcal{J}_1$ and $\mathcal{J} \neq \mathcal{J}_1$, then $\mathcal{J}$ is not universal. The paper ends with the result that the well known from literature Mycielski σ-ideal $\mathcal{J}_{\mathcal{M}}$ (J. Mycielski, *Some new ideals of sets on the real line*, Colloq. Math. 20, 71–77, 1960) is not universal.

Paper [?] is a chapter in the monograph *Traditional and present-day topics in real analysis* (Faculty of Mathematics and Computer Science, University of Łódź, 431–447, 2013) dedicated to Professor S. Lipiński. This paper is mostly a review. The first part contains the properties of topologies generated by the lower density operators on a measurable space $\langle X, \mathcal{S}, \mathcal{J} \rangle$ with the distinguished σ-ideal $\mathcal{J}$. The second part concerns the almost lower density operators on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ and the properties of topologies generated by them. The presented proofs come from the published papers of Authors. The main goal of this paper is to show some dual properties of density type topologies generated by the lower and almost lower density operators, as well as to notice differences between such topologies.

Paper [?] is a chapter in *Monograph on the occasion of 100th birthday anniversary of Zygmunt Zahorski* (Publisher: Silesian Technical University, Gliwice, 141–154, 2015). This paper contain the results related to the topologies generated by the lower and almost lower density operators associated with the sequences of closed, non-degenerated intervals convergent to zero. The first part of the paper concerns the topologies generated by the lower density operators on $\langle \mathbb{R}, \mathcal{B}a, \mathbb{K} \rangle$. It contains the results of papers published by the Authors and it is related to the properties of these topologies as well as properties of continuous functions relative to them. The second part of the paper presents the results of papers on topologies generated by the Φ_J-operators on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L} \rangle$. There are the results of J-density topologies generated both by an operator Φ_J being the lower density operator as well as being the almost lower density operator. The properties of $\mathcal{T}_{\Phi_J}$-continuous functions are also included.

Paper [?] is a chapter in the monograph *Modern Real Analysis* (Faculty of Mathematics and Computer Science, University of Łódz, 61–68, 2015). The main subject of this paper are considerations on density type topologies generated by the lower density operator dealing with some operator $\Phi : \tau_{nat} \to 2^{\mathbb{R}}$ with the following properties:

i) $\Phi(\emptyset) = \emptyset$ and $\Phi(\mathbb{R}) = \mathbb{R}$,

ii) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ for any $A, B \in \tau_{nat}$,

iii) $A \subset \Phi(A)$ for every $A \in \tau_{nat}$.

Putting, for $A \in \mathcal{B}a$

$$\Phi_r(A) = \Phi(G(A)),$$

where $G(A)$ is τ_{nat}-regular open and $A \triangle G(A) \in \mathbb{K}$, we get the lower density operator Φ_r on $\langle \mathbb{R}, \mathcal{B}a, \mathbb{K} \rangle$ generating topology $\mathcal{T}_{\Phi_r}$. The subject of paper [?] is the investigation of $\mathcal{T}_{\Phi_r}$-continuous functions and $\mathcal{T}_{\Phi_r}$-approximately continuous functions. $\mathcal{T}_{\Phi_r}$-approximate continuity of a function $f : \mathbb{R} \to \mathbb{R}$ at $x_0 \in \mathbb{R}$ means that there is a set $A \in \mathcal{B}a$ with $x_0 \in \Phi_r(A)$ and $f_{|A \cup \{x_0\}}$ is τ_{nat}-continuous. $\mathcal{T}_{\Phi_r}$-approximate continuity implies $\mathcal{T}_{\Phi_r}$-continuity. This paper contains the condition of the equivalence of these continuities. If Φ is the identity operator, then $\mathcal{T}_{\Phi_r}$-approximate continuity and $\mathcal{T}_{\Phi_r}$- continuity are equivalent.

4 About some scientific achievements before habilitation

While still studying, I was interested in some comparisons of the measure and the category. The excellent monograph J. C. Oxtoby *Measure and Category* (J. C. Oxtoby, *Measure and Category*, Springer-Verlag, New York, 1971) significantly contributed to such interest. Some fundamental examples of the properties of the measure and the category indicate both the richness and also the wealth of differences.

For new comparisons of the measure and the category, the great influence had the concept of convergence of the real $\mathcal{S}$-measurable functions with respect to a σ-ideal $\mathcal{J}$, defined in introduction, where $\langle X, \mathcal{S}, \mathcal{J} \rangle$ is a measurable space with the distinguished proper σ-ideal $\mathcal{J}$. Because of this new idea of convergence, it was possible to develop comparative research into the family of measurable functions in the Lebesgue sense and the family of functions having the Baire property related topologization of these families by the closure operation defined by convergence with respect to the appropriate σ-ideal. In the first case it is known that convergence with respect to measure is even metrizable while the space of functions having the Baire property is not topologizable by convergence with respect to the σ-ideal of the first category. This made me intensively deal with topologization of an $\mathcal{S}$-measurable family of functions through the closing operation defined by convergence with respect to a σ-ideal $\mathcal{J}$. Considering the family of real, measurable functions with respect to the σ-ideal $\mathcal{B} \triangle \mathcal{J}$, which was a part of my PhD dissertation, I found in paper [?], that for the Mycielski σ-ideal $\mathcal{J}_M$ the space of $\mathcal{B} \triangle \mathcal{J}_M$-measurable functions is not topologizable. In paper [?] I showed, a similar property for every totally imperfect σ-ideal. The examples of such ideals are abundant in literature. You can not, however, generate a totally imperfect σ-ideal from the totally imperfect σ-ideals. In paper [?], I found that there are σ-ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ totally imperfect and invariant with respect to the translation that the σ-ideal generated by $\mathcal{J}_1$ and $\mathcal{J}_2$ contain a non-empty perfect set.

In 1989, in the Banach Center, I learnt from the lectures of Professor J. Morgan, contained in the later published monograph *Point Set Theory* (Chapman & Marcel-Dekker

Inc., New York and Basel, 1990), that operating with the concept of the category base, we can often manage common properties for the measure and the category in the new uniform concept of the language of the category bases. A topological space or a space with σ-finite measure, or a measurable space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ with the distinguished σ-ideal $\mathcal{J}$ and such that the pair $\langle \mathcal{S}, \mathcal{J}\rangle$ meets c.c.c., are the most natural examples of category bases. For this reason, in paper [?], I studied the convergence of multi-indexes sequences of functions $\{f_{m,n}\}_{m,n\in\mathbb{N}}$ in the category bases. I observed that the convergence almost everywhere in a category base of all subsequences $\{f_{m_k,n_k}\}_{k\in\mathbb{N}}$ implies the convergence of the sequence $\{f_{m,n}\}_{m,n\in\mathbb{N}}$ almost everywhere. While investigating topologizations of the family of the measurable functions in a separable category base, I presented in paper [?], some local property that interferes with obtaining a topology. On this basis, I characterized T_1 topological spaces with a countable base, in which topologization of the Baire's functions is available. Because of this result I got in paper [?] the property describing no topologization of the family of $\mathcal{B}\triangle\mathcal{J}$-measurable functions in a Polish space when the σ-ideal $\mathcal{J}$ has a base composed of F_σ-sets. With the assumption that in any category base the sets of the power less than the power of the space belong to σ-ideal of the first category set that has a base of the power not exceeding the power of the space, I concluded in paper [?] the existence of a subset not being a Baire set. It implies the existence of a non-measurable set in the Lebesgue sense and a set not having the Baire property, respectively. I studied conditions under this aspect in paper [?], when any uncountable family of sets of the first category in a fixed category base, has a subfamily, which union is not a Baire set.

The concept of convergence of the sequence of functions with respect to a σ-ideal encouraged me to study the separability and compactness of the spaces of $\mathcal{S}$-measurable functions. In paper [?] a necessary and sufficient condition of separability such spaces I proved, among the other things. In paper [?], there is characterization the compactness of the family $\mathcal{S}$-measurable functions with value in a Hausdorff space which is a regular and Lindelöf space the diagonal of G_δ-type.

A certain, parallel topic of my interests, was the study of small systems of sets, initiated by B. Riečana in the paper *Subadditive measures and small systems* (Časopis

Pěst. Mat. 99 no. 3, 279–285, 1974). The small system estimates the size of the set by belonging to the appropriate family which is one of the elements of the decreasing sequence of families of sets. The intersection of the small system is a σ-ideal, which elements are treated as small sets. In the language of the small system I have expressed the theorem of Egoroff in paper [?] and in paper [?] the theorem of Lusin. We get from them the classic theorem of Egoroff and Lusin, when a small system is generated by a σ-finite measure. Paper [?] is providing necessary and sufficient condition for the compactness of the family of $\mathcal{S}$-measurable functions in the aspect of convergence with respect to the small system. It is generalization of Fréchet theorem on the compactness of measurable functions in the sense of convergence with respect to measure.

Paper [?] concerns the study of σ-ideals with the Borel base in an uncountable Polish space X. With every proper σ-ideal $\mathcal{J}$ in the space X with the Borel and G_δ base we can associate a σ-ideal $\mathcal{J}_1 \subset \mathcal{J}$ such that every element of $\mathcal{J}_1$ has a superset F_σ-type from the family $\mathcal{J}$. Such σ-ideal $\mathcal{J}_1$ is comparable to the family of σ-porous sets in uncountable Polish space. It is proved that for σ-ideal $\mathcal{J}$ in an uncountable Polish space there is a sequence of ω_1 length of nowhere dense, pairwise disjoint G_δ-sets from the family $\mathcal{J} \setminus \mathcal{J}_1$. If $\mathcal{J} = \mathbb{L}$, there are 2^{ω_0} sets pairwise disjoint G_δ-sets from the family $\mathbb{L} \setminus \mathbb{L}_1$. The similar result was known for the σ-porous sets. Namely, J. Tkadlec in the paper *Construction of some non-σ-porous sets on real line* (Real Anal. Exchange 9, 473–482, 1983) proved that for an arbitrary set A of the first category there is a prefect set B, which is not σ-porous and disjoint with A.

In paper [?], influenced by a stay at the seminar of Professor C. Weila in Michigan State University (USA), who is the specialist on the Peano derivative, I investigated the Peano derivative for a function with the Baire property. I received the result that if a function $f : \mathbb{R} \to \mathbb{R}$ has the Baire property, then for every $k \in \mathbb{N}$ the set $E_k = \{x \in \mathbb{R} : f_{(k)}(x) \text{ exists}\} \in \mathcal{B}a$, where $f_{(k)}$ is k-th coefficient in the definition of the Peano derivative. I introduced in this paper the concept of the $\mathcal{I}$-approximative Peano derivative. We shall say that a function $f : \mathbb{R} \to \mathbb{R}$ defined in a neighbourhood of x has the $\mathcal{I}$-approximative Peano derivative of the k-th order, if there exists the numbers $f_{(1)\mathcal{I}-ap}(x), \ldots, f_{(k)\mathcal{I}-ap}(x)$ and a set $E \in \mathcal{B}a$ such that x is an $\mathcal{I}$-density point of E

and

$$\lim_{\substack{t\to 0\\ t\in E}} \frac{f(x+t)-f(x)-\sum\limits_{i=1}^{k}\frac{t^i}{i!}f_{(i)\mathcal{I}-ap}(x)}{t^k}=0.$$

I proved that if $f:\mathbb{R}\to\mathbb{R}$ has the Baire property, then for every $k\in\mathbb{N}$ the set $E_k=\{x\in\mathbb{R}: f_{(k)\mathcal{I}-ap}(x) \text{ exists}\}\in\mathcal{B}a$ and the function $f_{(k)\mathcal{I}-ap}$ has the Baire property.

In my first published paper [**?**], using the property of S. Piccard of the non-empty interior of the algebraic sum two sets of the second category with the Baire property, I investigated the form of each of the sets of countable family consisting of the decomposition of the space $\mathbb{R}^n$ assuming that every set of this family has the Baire property and fulfills the condition of additivity. This is a generalization of M. Kuczma theorem obtained for two sets A and $\mathbb{R}^n\setminus A$.

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[56] J. Hejduk, R. Wiertelak, *"On some properties of J-appropximately continuous functions"*, Math. Slovaca 67 no. 6 (2017), 1–10 (Journal Impact Factor 2017: 0,314).

[57] J. Hejduk, A. Loranty, *"On abstract and almost abstract density topologies"*, Acta Math. Hungarica 155 no. 2 (2018), 228–240 (Journal Impact Factor 2017: 0,481).

[58] J. Hejduk, S. Lindner, A. Loranty, *"On lower density type operators and topologies generated by them"*, Filomat 32 no. 14 (2018), 4949–4957 (Journal Impact Factor 2017: 0,635).

[59] J. Hejduk, R. Wiertelak, *"On topologies related to the extension of Lebesgue measure"*, to appear in Georgian Math. J. https://doi.org/10.1515/gmj-2018-0050 (Journal Impact Factor 2017: 0,482).

5 List of publications with the number of citations

The source concerns three databases: Mathscinet, Web of Science and Google Scholar.
- Mathscinet database provides 58 citations by 32 authors, including 7 self-citations. The Hirsch index according to this base is 5.

- The Web of Sience database provides 71 citations by 32 authors, including 10 self-citations. The Hirsch indexaccording to this base is 5.
- Google Scholar database provides 179 citations by 47 authors. The Hirsch index according to this base is 8.

In addition, the Web of Science Database allows to generate a list of citations by individual authors (the number of citations in the brackets is marked):

- M. Balcerzak (2)
- A. Bartoszewicz (1)
- K. Czudek (1)
- M. Filipczak (5)
- T. Filipczak (4)
- K. Flak (1)
- M. Grinc (1)
- J. Hejduk (10)
- G. Horbaczewska (5)
- G. Ivanova (1)
- A. Karasińska (1)
- A. S. Kechris (1)
- E. Kotlicka (1)
- A. Kowalski (1)
- A. Kwela (1)
- J. A. Larson (1)
- S. Lindner (4)
- A. Loranty (4)
- N. D. Macheras (1)
- N. Mrozek (1)
- R. J. Pawlak (1)
- S. Shelah (1)
- S. Solecki (1)
- J. Steprans (1)
- W. Strauss (1)
- F. Strobin (2)
- M. Terepeta (4)
- E. Wajch (1)
- R. Wiertelak (7)
- W. Wilczyński (2)
- W. Wojdowski (2)
- W. Woloszyn (1)

From the information published in Web of Science, it appears that the publications are cited in the articles published in the following magazines/publishing series:

- Acta Mathematica Hungarica
- Archive For Mathematical Logic
- Bulletin of the American Mathematical Society
- Bulletin of the Australian Mathematical Society
- Central European Journal of Mathematics
- Czechoslovak Mathematical Journal
- Filomat
- Georgian Mathematical Journal
- Israel Journal of Mathematics

- Journal of Symbolic Logic
- Lithuanian Mathematical Journal
- Mathematica Slovaca
- Mathematical Logic Quarterly
- Positivity
- Publicationes Mathematicae Debrecen
- Real Function 09 Measure Integration Harmonic Analysis Topology and Mathematical Economics
- Real Function 08 Functional Equations Measure Integration and Harmonic Analysis Topology
- Real Functions 10
- Real Function 15 Measure Theory Real Functions General Topology
- Tatra Mountains Mathematical Publications
- Topology and its Aplications

There are below the lists of papers with the citations numbers with respect to the particular base.

5.1 Citations with respect to the Mathscinet base

List of publications with the number of citations with respect to the Mathscinet base:

i) M. Filipczak, J. Hejduk, *"On topologies associated with Lebesgue measure"*, Tatra Mt. Math. Publ. 28 part II (2004), 187–197 – the paper cited 13 times.

ii) M. Filipczak, T. Filipczak, J. Hejduk, *"On the comparison of the density type topologies"*, Atti. Semin. Mat. fis. Univ. Modena Reggio Emilia 52 no. 1 (2004), 37–46 – the paper cited 7 times.

iii) M. Balcerzak, J. Hejduk, *"Density topologies for products of σ-ideals"*, Real Analysis Exchange 20 (1) (1994-95), 163–177 – the paper cited 6 times.

iv) J. Hejduk, R. Wiertelak, *"On the generalization of density topologies on the real line"*, Math. Slovaca 64 no. 5 (2014), 1267–1276 – the paper cited 6 times.

v) J. Hejduk, *"On topologies with respect to invariant σ- ideals"*, Journal of Applied Analysis, vol.8, No.2 (2002), 201–219 – the paper cited 5 times.

vi) J. Hejduk, R. Wiertelak, *"On the abstract density topologies generated by lower and almost lower density operators"*, rozdział w monografii: Traditional and present - day topics in real analysis, Faculty of Mathematics and Computer Science, University of Łódz, 2013, 431–447 – the paper cited 4 times.

vii) M. Balcerzak, J. E. Baumgartner, J. Hejduk, *"On certain σ-ideal of sets"*, Real Analysis Exchange 14 (1988-89), 447–453 – the paper cited 2 times.

viii) M. Filipczak, J. Hejduk, W. Wilczyński, *"On the homeomorphisms of the density type topologies"*, Comment. Math., Prace Mat. 45 no. 2 (2005), 151–159 – the paper cited 2 times.

ix) J. Hejduk, *"Remarks on the density topologies generated by functions"*, Słupskie Prace Matematyczno - Fizyczne No. 4 (2007), 39–46 – the paper cited 2 times.

x) J. Hejduk, *"One more difference between measure and category"*, Tatra Mt. Math. Publ. 49 (2011), 9–15 – the paper cited 2 times.

xi) J. Hejduk, *"Convergence with respect to the Mycielski σ-ideal"*, Demonstratio Math. 22 (1) (1989), 43–50 – the paper cited 1 time.

xii) J. Hejduk, S. Lindner, *"On the Hashimoto topology with respect to an extension of the Lebesgue measure"*, Tatra Mt. Math. Publ. 24 (2002), 147–153 – the paper cited 1 time.

xiii) J. Hejduk, *"On the density topologies generated by functions"*, Tatra Mt. Math. Publ. 40 (2008), 133–141 – the paper cited 1 time.

xiv) J. Hejduk, A. Loranty, *"Remarks on the topologies in the Lebesgue measurable sets"*, Demonstratio Math. vol. 45 (3) (2012), 655–663 – the paper cited 1 time.

xv) J. Hejduk, *"On the regularity of topologies in the family of sets having the Baire property"*, Filomat 27 no. 7 (2013), 1291–1295 – the paper cited 1 time.

xvi) J. Hejduk, A. Loranty, R. Wiertelak, *"On J-continuous functions"*, Tatra Mt. Math. Publ. 65 (2016), 49–59 – the paper cited 1 time.

xvii) J. Hejduk, R. Wiertelak, W. Wojdowski, *"On semiregularization of some abstract density topologies involving sets having the Baire property"*, Tatra Mt. Math. Publ. 65 (2016), 37–48 – the paper cited 1 time.

5.2 Citations with respect to the Web of Science base

List of publications with the number of citations with respect to the Web of Science base:

i) M. Filipczak, J. Hejduk, *"On topologies associated with Lebesgue measure"*, Tatra Mt. Math. Publ. 28 (2004), part II, 187–197 – cited in 15 papers.

ii) M. Filipczak, T. Filipczak, J. Hejduk, *"On the comparison of the density type topologies"*, Atti. Semin. Mat. fis. Univ. Modena Reggio Emilia 52 no. 1 (2004), 37–46 – cited in 7 papers.

iii) J. Hejduk, R. Wiertelak, *"On the generalization of density topologies on the real line"*, Math. Slovaca 64 no. 5 (2014), 1267–1276 – cited in 8 papers.

iv) J. Hejduk, G. Horbaczewska, *"On I-density topology with respect to a fixed sequence"*, Reports on Real Analysis, Słupsk (2003), 78–85 – cited in 5 papers.

v) J. Hejduk, R. Wiertelak, *"On the abstract density topologies generated by lower and almost lower density operators"*, rozdział w monografii: Traditional and present - day topics in real analysis, Faculty of Mathematics and Computer Science, University of Łódz, 2013, 431–447 – cited in 5 papers.

vi) M. Balcerzak, J. Hejduk, *"Density topologies for products of σ-ideals"*, Real Analysis Exchange 20 (1) (1994-95), 163–177 – cited in 4 papers.

vii) J. Hejduk, A. Loranty, *"On the lower and semi-lower density operators"*, Georgian Math. J. vol. 14 No. 4 (2007), 661–671 – cited in 4 papers.

viii) J. Hejduk, *"One more difference between measure and category"*, Tatra Mt. Math. Publ. 49 (2011), 9–15 – cited in 4 papers.

ix) J. Hejduk, *"On the regularity of topologies in the family of sets having the Baire property"*, Filomat 27 no. 7 (2013), 1291–1295 – cited in 4 papers.

x) M. Balcerzak, J. E. Baumgartner, J. Hejduk, *"On certain σ-ideal of sets"*, Real Analysis Exchange 14 (1988-89), 447–453 – cited in 3 papers.

xi) J. Hejduk, *"On the density topologies generated by functions"*, Tatra Mt. Math. Publ. 40 (2008), 133–141 – cited in 3 papers.

xii) M. Filipczak, J. Hejduk, W. Wilczyński, *"On the homeomorphisms of the density type topologies"*, Comment. Math., Prace Mat. 45 no. 2 (2005), 151–159 – cited in 3 papers.

xiii) J. Hejduk, *"On the abstract density topologies"*, selected papers of the 2010 International Conference on Topology and its Application, (2012), 79–85 – cited in 3 papers.

xiv) J. Hejduk, R. Wiertelak, A. Loranty, *"On J-continuous functions"*, Tatra Mt. Math. Publ. 65 (2016), 49–59 – cited in 3 papers.

xv) J. Hejduk, A. Loranty, *"Remarks on the topologies in the Lebesgue measurable sets"*, Demonstratio Math. vol. 45 (3) (2012), 655–663 – cited in 2 papers.

xvi) J. Hejduk, R. Wiertelak, A. Loranty, *"J-approximately continuous functions"*, Tatra Mt. Math. Publ. 62 (2015), 45–55 – cited in 2 papers.

xvii) J. Hejduk, A. B. Kharazishvili, *"On density points with respect to von Neumann topology"*, Real Analysis Exchange 21 (1) (1995-96), 278–291 – cited in 1 paper.

xviii) J. Hejduk, *"Universal sequences in the space of real measurable functions"*, Scientific Bulletin of Łódź Technical University 21 (1989), 75–85 – cited in 1 paper.

xix) J. Hejduk, *"Some properties of topological σ-ideals"*, Demonstratio Math. 22 (4) (1989), 1183–1189 – cited in 1 paper.

xx) J. Hejduk, E. Wajch, *"Compactness in the sense of the convergence with respect to a small system"*, Math. Slovaca 3 (1989), 267–275 – cited in 1 paper.

xxi) J. Hejduk, *"On Lusin theorem in the aspect of small system"*, Demonstratio Math. 28 (1) (1995), 107–110 – cited in 1 paper.

xxii) J. Hejduk, *"On the density topology with respect to an extension of the Lebesgue measure"*, Real Analysis Exchange 21 (2) (1995-96), 811–816 – cited in 1 paper.

xxiii) J. Hejduk, *"Density topologies with respect to invariant σ-ideals"*, Wydawnictwo Uniwersytetu Łódzkiego, Łódź 1997 – cited in 1 paper.

xxiv) J. Hejduk, S. Lindner, *"On the Hashimoto topology with respect to an extension of the Lebesgue measure"*, Tatra Mt. Math. Publ. 24 (2002), 147–153 – cited in 1 paper.

xxv) J. Hejduk, K. Flak, *"On the topologies generated by some operators"*, Cent. European J. of Math. no. 2 (2013), 349–357 – cited in 1 paper.

xxvi) J. Hejduk, R. Wiertelak, W. Wojdowski, *" On semiregularization of some abstract density topologies involving sets having the Baire property"*, Tatra Mt. Math. Publ. 65 (2016), 37–48 – cited in 1 paper.

5.3 Citations with respect to the Google Scholar base

List of publications with the number of citations with respect to the Google Scholar base:

i) M. Filipczak, J. Hejduk, *"On topologies associated with Lebesgue measure"*, Tatra Mt. Math. Publ. 28 (2004), part II, 187–197 – the paper cited 32 times.

ii) M. Filipczak, T. Filipczak, J. Hejduk, *"On the comparison of the density type topologies"*, Atti. Semin. Mat. fis. Univ. Modena Reggio Emilia 52 no. 1 (2004), 37–46 – the paper cited 20 times.

iii) J. Hejduk, R. Wiertelak, *"On the generalization of density topologies on the real line"*, Math. Slovaca 64 no. 5 (2014), 1267–1276 – the paper cited 12 times.

iv) J. Hejduk, R. Wiertelak, *"On the abstract density topologies generated by lower and almost lower density operators"*, rozdział w monografii: Traditional and present - day topics in real analysis, Faculty of Mathematics and Computer Science, University of Łódz, 2013, 431–447 – the paper cited 11 times.

v) J. Hejduk, G. Horbaczewska, *"On I-density topology with respect to a fixed sequence"*, Reports on Real Analysis, Słupsk (2003), 78–85 – the paper cited 10 times.

vi) M. Balcerzak, J. Hejduk, *"Density topologies for products of σ-ideals"*, Real Analysis Exchange 20 (1) (1994-95), 163–177 – the paper cited 10 times.

vii) J. Hejduk, *"On topologies with respect to invariant σ- ideals"*, Journal of Applied Analysis, vol. 8, No. 2 (2002), 201–219 – the paper cited 8 times.

viii) J. Hejduk, A. B. Kharazishvili, *"On density points with respect to von Neumann topology"*, Real Analysis Exchange 21 (1) (1995-96), 278–291 – the paper cited 8 times.

ix) J. Hejduk, A. Loranty, *"On the lower and semi-lower density operators"*, Georgian Math. J. vol. 14 No. 4 (2007), 661–671 – the paper cited 6 times.

x) M. Filipczak, J. Hejduk, W. Wilczyński, *"On the homeomorphisms of the density type topologies"*, Comment. Math., Prace Mat. 45 no. 2 (2005), 151–159 – the paper cited 6 times.

xi) M. Balcerzak, J. E. Baumgartner, J. Hejduk, *"On certain σ-ideal of sets"*, Real Analysis Exchange 14 (1988-89), 447–453 – the paper cited 6 times.

xii) J. Hejduk, *"One more difference between measure and category"*, Tatra Mt. Math. Publ. 49 (2011), 9–15 – the paper cited 5 times.

xiii) J. Hejduk, A. Loranty, R. Wiertelak, *"On J-continuous functions"*, Tatra Mt. Math. Publ. 65 (2016), 49–59 – the paper cited 4 times.

xiv) J. Hejduk, *"On the regularity of topologies in the family of sets having the Baire property"*, Filomat 27 no. 7 (2013), 1291–1295 – the paper cited 4 times.

xv) J. Hejduk, *"On the density topologies generated by functions"*, Tatra Mt. Math. Publ. 40 (2008), 133–141 – the paper cited 4 times.

xvi) J. Hejduk, A. Loranty, R. Wiertelak, *"J-approximately continuous functions"*, Tatra Mt. Math. Publ. 62 (2015), 45–55 – the paper cited 3 times.

xvii) J. Hejduk, *"On the density topology with respect to an extension of the Lebesgue measure"*, Real Analysis Exchange 21 (2) (1995-96), 811–816 – the paper cited 3 times.

xviii) J. Hejduk, A. Loranty, *"Remarks on the topologies in the Lebesgue measurable sets"*, Demonstratio Math. vol. 45 (3) (2012), 655–663 – the paper cited 2 times.

xix) J. Hejduk, *"Some properties of the density topology with respect to an extension of the Lebesgue measure"*, Math. Pannon. 9 no. 2 (1998), 173–190 – the paper cited 2 times.

xx) J. Hejduk, *"Non - Baire sets in category bases"*, Real Analysis Exchange 18 (2) (1993-93), 448–453 – the paper cited 2 times.

xxi) J. Hejduk, E. Wajch, *"Compactness in the sense of the convergence with respect to a small system"*, Math. Slovaca 3 (1989), 267–275 – the paper cited 2 times.

xxii) J. Hejduk, *"Convergence with respect to the Mycielski σ-ideal"*, Demonstratio Math. 22 (1) (1989), 43–50 – the paper cited 2 times.

xxiii) J. Hejduk, A. Loranty, *"On abstract and almost abstract density topologies"*, Acta Math. Hungarica 155 no.2 (2018), 228–240 – the paper cited 1 time.

xxiv) J. Hejduk, R. Wiertelak, *"On some properties of J-appropximately continuous functions"*, Math. Slovaca 67 no. 6 (2017), 1–10 – the paper cited 1 time.

xxv) J. Hejduk, R. Wiertelak, W. Wojdowski, *" On semiregularization of some abstract density topologies involving sets having the Baire property"*, Tatra Mt. Math. Publ. 65 (2016), 37–48 – the paper cited 1 time.

xxvi) J. Hejduk, *"On topologies in the family of sets with the Baire property"*, Georgian Math. J no. 2 (2015), 243–250 – the paper cited 1 time.

xxvii) K. Flak, J. Hejduk, *"On equivalence of topological and restrictional continuity"*, rozdział w monografii: Modern Real Analysis, Faculty of Mathematics and Computer Science, University of Łódz, 2015, 61–68 – the paper cited 1 time.

xxviii) K. Flak, J. Hejduk, *"On the topologies generated by some operators"*, Cent. European J. of Math. no. 2 (2013), 349–357 – the paper cited 1 time.

xxix) J. Hejduk, S. Lindner, *"On the Hashimoto topology with respect to an extension of the Lebesgue measure"*, Tatra Mt. Math. Publ. 24 (2002), 147–153 – the paper cited 1 time.

xxx) J. Hejduk, *"On Lusin theorem in the aspect of small system"*, Demonstratio Math. 28 (1) (1995), 107–110 – the paper cited 1 time.

xxxi) J. Hejduk, *"Convergence with respect to the σ-ideal of meager sets in separable category bases"*, Demonstratio Math. 28 (3) (1995), 619–623 – the paper cited 1 time.

xxxii) J. Hejduk, *"Convergence with respect to some σ-ideals"*, Univ.u Novom Sadu Zb. Rad. Prirod. - Mat.Fak. Ser. Mat. 21 (1) (1991), 157–164 – the paper cited 1 time.

xxxiii) J. Hejduk, *"Some properties of topological σ-ideals"*, Demonstratio Math. 22 (4) (1989), 1183–1189 – the paper cited 1 time.

5.4 Citations in the monographs

- In the monograph of B. Riečan, T. Neubrun, *"Integral, measure and ordering"*, Kluver Academic Publishers, Ister Science, Bratislava, 1997 cited papers:

i) J. Hejduk, E. Wajch, *"Compactness in the sense of the convergence with respect to a small system"*, Math. Slovaca 3 (1989), 267–275 (Journal Impact Factor 2017: 0,314).

ii) J. Hejduk, *"Some remarks on Lusin theorem in the abstract sense"*, XXXV semester in Banach–Center, December 1990, 2–8.

iii) J. Hejduk, *"On Lusin theorem in the aspect of small system"*, Demonstratio Math. 28 (1) (1995), 107–110.

- In the monograph of A. B. Kharazishvili, *"Nonmeasurable sets and functions"*, Amsterdam, Elsvier 2004;

cited paper:

J. Hejduk, A. B. Kharazishvili, *"On density points with respect to von Neumann topology"*, Real Analysis Exchange 21 (1) (1995-96), 278–291.

6 Achievements in the field of scientific care and education of the young staff

Completed proceedings for granting the doctoral degree in which I was a supervisor:

1. dr Sebastian Lindner, University of Łódź, 2003. The title of the PhD dissertation: *On certain properties of measure obtained by variation.*

2. dr Anna Loranty, University of Łódź, 2005. The title of the PhD dissertation: *On density topologies generated by the sequences of non-decreasing and unbounded sequences of positive reals.*

3. dr Renata Wiertelak, University of Łódź, 2008. The title of the PhD dissertation: *On the density topologies generated by the sequences of intervals convergent to zero.*

4. dr Magdalena Górajska, University of Łódź, 2011. The title of the PhD dissertation: *On the point density topologies.*

Open doctoral truck in which I am participating as a supervisor:

1. mgr Mikołaj Widzibor, University of Łódź, the doctoral truck opened in February 2019 roku. The title of the PhD dissertation: *On topologies generated by the regular sequences of measurable sets.*

Reviews of the PhD dissertations:

1. dr Monika Marciniak, University of Łódź, 2000. The title of the PhD dissertation: *The properties of functions with respect to the system of paths.*
2. dr Joanna Rzepecka, Technical University of Łódź, 2003. The title of the PhD dissertation: *Marczewski-Burstin representation of some algebras and ideals.*
3. dr Monika Potyrała, Technical University of Łódź, 2006. The title of the PhD dissertation: *On some modifications and properties of Birkhoff integral.*
4. dr Marta Frankowska, University of Gdańsk, 2013. The title of the PhD dissertation: *On the selected properties of topologies with respect to the some ideal.*

In the years 2011–2016 I was the scientific supervisor of Mikołaj Widzibor at the inter-branch Studies of Natural Sciences at the University of Łódź.

Since 2012 I have been the Member of the Program Council of the Inter-faculty studies in mathematics and natural sciences.

I was the supervisor of over 35 Masters thesis and the reviewer of over 75 Masters thesis. I was three times the member of Habilitation Committee and five times the secretary of the habilitation trucks.

7 On the activity promoting Science

In the years 2011-2018, as an apart of the Erasmus programs, I spent one week stays at the University of Joannina in Greece, University of Mersin and Cukurova University in Adana(Turkey), University of Santiago de Compostela (Spain), University of Grenada (Spain), Commerce University of Instanbul (Turkey), University of Palermo (Italy) and Lucjan Blaga University of Sibiu (Romania). This year I have been qualified for a stay in Goce Delcev State University in the Republic of Northen Macedonia. During my visits, additionally to the formal lectures and seminars, I presented in informal meetings the ideas oof the contributions of Polish mathematicians to breaking Enigma machine and about the phenomena of the Polish Mathematical School before the second World War. At the University in Instanbul, Ankara, Palermo, and Sibiu I included in my program of lectures the formal presentations:

- *On the contribution of the Polish Mathematicians into the breaking Enigma machine*
- *On the Polish Mathematical School*

I have also participated in the Festival of Science, Technology and Arts organized by University of Łódź delivering the lectures:

- *On mathematical inquisitivity on the basis of generalization the inequality of Bernoulli?*
- *Why do we appreciate non-standard solutions of mathematical problems?*
- *On the selection of mandates after elections*
- *About the selection of mandates to the Polish Parlament*

In 2017, I gave the inauguration lecture at the Faculty of Mathematics and Computer Science of Łódz University entitled:

From the ballot box to the mandate in the Polish Parlament

In 2018 I presented the similar lecture at the meeting of Łódź Scientific Society.

In March 2019, I made a commentary (in the form of a film broadcast on You Tube of the University of Lodz) about breaking the Enigma code by Polish mathematicians in the context of the issued polish translation of the Dermont Turing monograph "XYZ. The real story of how Enigma was broken." The recording took place in within the project "UŁ comments", in which lecturers and graduates of the University comment the current events in Poland and in the world.

8 On the organizational activities

From 1984 to 2002, I had been the Secretary of the Journal Folia Mathematica.

From 2002 to 2012, I had been the Editor of Journal Folia Mathematica.

From 2013, I have become a member of the Publishing Council of the University of Łódź.

In 2015, I was appointed by the Rector UŁ as a member of the Scientific Council of the publishing series "Publications of young scientists" for the period 2015-2016.

In 1989, I was the Secretary during the semester devoted to Real Analysis in the Banach Center in Warsaw.

In 1994, I was a co-organizer of the Polish-American Workshop held in Łódź co-organized with the Chair of Real Functions of the University of Łódź.

In 1999, I was a co-organizer of the XXIII Summer Symposium in Real Analysis "The Lodz Symposium".

In the term of 2002–2005 I was the Associate Dean for Academic and Student Affairs at the Faculty of Mathematics of the University of Łódź.

In the term of 2005–2008 I was the Associate Dean for Academic and Student Affairs at the Faculty of Mathematics and Computer Science of the University of Łódź.

From 2008 to the present I am a representative of the Faculty to the Senat of the University of Łódź.

In the period 2008–2016 I was the member of the Statute Counsil.

I am a member of the Business Council established in 2016 at the Faculty of Mathematics and Computer Science cooperating with 29 companies and organizing student internships.

In 2018, I was admitted to the Łódź Scientific Society.

In March 2019, I was appointed by the Rector of the University of Lodz to represent the University in the Scientific Council of the Mianowski Fund in Warsaw.