Report

Dariusz Idczak

Contents

1 Scientific achievements

1.1 Publications in scientific journals

[1] D. Idczak, *Applications of the fixed point theorem to problems of controllability*, Bulletin de la Societe des Sciences et des Lettres de Lodz, vol. XXXIX.3, no. 57 (1989), 1-7.

[2] D. Idczak, S. Walczak, *On the controllability of nonlinear Goursat systems*, Optimization 23 (1992), 91-98.

[3] D. Idczak, K. Kibalczyc, S. Walczak, *On some optimal control problem for two-dimensional continuous systems*, Bull. Polish Acad. Sci. Tech. Sci.41 (4) (1993), 371-379.

[4] D. Idczak, *Nonlinear Goursat-Darboux problem and its optimization*, Nonlinear Vibration Problems 25 (1993), 143-157.

[5] D. Idczak, S. Walczak, *Optimal control hyperbolic systems with bounded variations of controls*, Lecture Notes in Pure and Applied Mathematics 160 (1994), 159-171.

[6] D. Idczak, *Functions of several variables of finite variation and their differentiability*, Annales Polonici Mathematici LX.1 (1994), 47-56.

[7] D. Idczak, S. Walczak, *On Helly's theorem for functions of several variables and its applications to variational problems*, Optimization 30 (1994), 331-343.

[8] D. Idczak, K. Kibalczyc, S. Walczak, *On an optimization problem with cost of rapid variation of control*, J. Austral. Math. Soc. Ser. B 36 (1994), 117-131.

[9] D. Idczak, S. Walczak, *On the existence of the Caratheodory solutions for some boundary value problems*, Rocky Mountain J. Math. 24 (1) (1994), 1-13.

[10] D. Idczak, *Optimal control of a continuous n-D Roesser model*, Bull. Polish Acad. Sci. Tech. Sci. 43 (2) (1995), 227-234.

[11] D. Idczak, J. Matula, S. Walczak, *The existence of an optimal solution for two-dimensional continuous control systems*, Bull. Polish Acad. Sci. Tech. Sci. 43 (2) (1995), 235-245.

[12] D. Idczak, *The generalization of the Du Bois-Reymond lemma for functions of two variables to the case of partial derivatives of any order*, Topology in Nonlinear Analysis, Banach Center Publications 35 (1996), 221-236.

[13] D. Idczak, *Necessary optimality conditions for a nonlinear continuous n-D Roesser model*, Mathematics and Computers in Simulation 41 (1996), 87-94.

[14] D. Idczak, *M-periodic problem of order 2k*, Topological Methods in Nonlinear Analysis 11 (1998), 169-185.

[15] D. Idczak, *Optimal control of a coercive Dirichlet problem*, SIAM J. Control Optim. 36 (4) (1998), 1250-1267.

[16] D. Idczak, *Distributional derivatives of functions of two variables of finite variation and their application to an impulsive hyperbolic equation*, Czechoslovak Mathematical Journal 48 (123) (1998), 145-171.

[17] D. Idczak, S. Walczak, *On the existence of a solution for some distributed optimal control hyperbolic system*, Internat. J. Math. & Math. Sci., vol. 23, no. 5 (2000), 297-311.

[18] D. Idczak, *Stability in semilinear problems*, J. Differential Equations 162 (2000), 64-90.

[19] D. Idczak, S. Walczak, *Existence of an optimal solution for a continuous Roesser problem with a terminal condition*, K.Gałkowski and J. Wood (Eds.): Multidimensional Signals, Circuits & Systems, Taylor & Francis, London, New York (2001), 183-189.

[20] D. Idczak, *O pewnych problemach wariacyjnych dla równań różniczkowych zwyczajnych z warunkami brzegowymi typu Dirichleta i okresowymi oraz ich zastosowaniu*, Wydawnictwo Uniwersytetu Łódzkiego, Łódź, 2000 (habilitation dissertation, based on the works [14], [15], [18] and containing some extensions of results obtained in the above-mentioned works).

[21] D. Idczak, A. Rogowski, *On a generalization of Krasnoselskii's theorem*, J. Austral. Math. Soc. 72 (2002), 389-394.

[22] D. Idczak, *Bang-bang principle for linear and nonlinear Goursat-Darboux problems*, Int. J. Control 76 (11) (2003), 1089-1094.

[23] D. Idczak, M. Majewski, S. Walczak, *Stability analysis of solutions to an optimal control problem associated with a Goursat-Darboux problem*, K. Gałkowski, R. W. Longman, E. Rogers (Eds.): Multidimensional Systems nD and Iterative Learning Control, Int. J. Appl. Math. Comput. Sci. 13 (1) (2003), 29-44.

[24] D. Idczak, S. Walczak, *On some properties of Goursat-Darboux systems with distributed and boundary controls*, Int. J. Control 77 (9) (2004), 837-846.

[25] D. Idczak, M. Majewski, *Bang-bang controls and piecewise constant ones for continuous Roesser system*, Multidimensional Systems and Signal Processing 17 (2006), 243-255.

[26] D. Idczak, *Maximum principle for optimal control of two-directionally continuous linear repetitive processes*, E. Zerz and K. Galkowski (Eds.): Recent Advances in Multidimensional Systems and Signals, Multidimensional Systems and Signal Processing 19 (2008), 411-423.

[27] D. Idczak, *Approximative piecewise constant bang-bang principle for differential repetitive processes*, Int. J. Control 82 (5) (2009), 910-917.

[28] D. Idczak, S. Walczak, *An asymptotical stability in variational sense for second order systems*, Advanced Nonlinear Studies 10 (2010), 161-174.

[29] D. Idczak, M. Majewski, *A generalization of the Poincare-Miranda theorem with an application to the controllability of nonlinear repetitive processes*, K. Gałkowski and E. Rogers (Eds.): Recent Developments on Multidimensional Systems, Control and Signals – Theory and Applications, ASIAN J. Control 12 (2) (2010), 1-9.

[30] D. Idczak, R. Kamocki, *Optimal control systems of second order with infinite time horizon – maximum principle*, Dr. Sc. S. Cakaj (Ed.): Modeling, Simulation and Optimization – Tolerance and Optimal Control, INTECH, Vukovar, 2010.

[31] D. Idczak, S. Walczak, *Optimal control systems of second order with infinite time horizon – existence of solutions*, Journal of Optimization Theory and Applications 147 (2010), 205-222.

[32] D. Idczak, M. Majewski, *Existence of optimal solutions of two-directionally continuous linear repetitive processes*, Multidimensional Systems and Signal Processing 23 (2012), 155-162, DOI: 10.1007/s11045-010-0105-4; published online: 7.04.2010.

[33] D. Idczak, R. Kamocki, *On the existence and uniqueness and formula for the solution of R-L fractional Cauchy problem in* $\mathbb{R}^n$, Fractional Calculus and Applied Analysis 14 (4) (2011), 538-553.

[34] D. Idczak, S. Walczak, *A fractional imbedding theorem*, Fractional Calculus and Applied Analysis 15 (3) (2012), 418-425.

[35] D. Idczak, A. Skowron, S. Walczak, *On the diffeomorphisms between Banach and Hilbert spaces*, Advanced Nonlinear Studies 12 (2012), 89–100.

[36] D. Idczak, M. Majewski, *Fractional fundamental lemma of order* $\alpha \in (n - 1/2, n)$ *with* $n \in \mathbb{N}$, $n \geq 2$, Dynamic Systems and Applications 21 (2012), 251-268.

[37] D. Idczak, A. Skowron, S. Walczak, *Sensitivity of a fractional integro-differential Cauchy problem of Volterra type*, Abstract and Applied Analysis, vol. 2013, Article ID 129478, 8 pages, http://dx.doi.org/10.1155/2013/129478.

[38] D. Idczak, R. Kamocki, *Fractional differential repetitive processes with Riemann-Liouville and Caputo derivatives*, Multidimensional Systems and Signal Processing 26 (2015), 193-206, DOI 10.1007/s11045-013-0249-0, published online: 25.09.2013.

[39] D. Idczak, S. Walczak, *Fractional Sobolev spaces via Riemann-Liouville derivatives,* Journal of Function Spaces and Applications, vol. 2013, Article ID 128043, 15 pages, http://dx.doi.org/10.1155/2013/128043.

[40] D. Idczak, A. Skowron, S. Walczak, *On a class of diffeomorphisms defined by integro-differential operators*, Calculus of Variations and PDEs, Banach Center Publications 101 (2014), 77-86.

[41] D. Idczak, *A global implicit function theorem and its applications to functional equations*, Discrete and Continuous Dynamical Systems, Series B 19 (8) (2014), 2549-2556.

[42] D. Idczak, S. Walczak, *Optimization of a fractional Mayer problem – existence of solutions, maximum principle, gradient methods*, Opuscula Mathematica 34 (4) (2014), 763-775.

[43] L. Bourdin, D. Idczak, *A fractional fundamental lemma and fractional integration by parts formula - Applications to critical points of Bolza functionals and to linear boundary value problems*, Advances in Differential Equations 20 (3-4) (2015), 213-232.

[44] D. Idczak, R. Kamocki, M. Majewski, S. Walczak, *Existence of optimal solutions to Lagrange problems for Roesser type systems of the first and fractional orders*, Applied Mathematics and Computation 266 (2015), 809-819.

[45] D. Idczak, S. Walczak, *Application of a global implicit function theorem to a general fractional integro-differential system of Volterra type*, Journal of Integral Equations and Applications 27 (4) (2015), 521-554.

[46] D. Idczak, S. Walczak, *On a linear-quadratic problem with Caputo derivative*, Opuscula Mathematica 36 (1) (2016), 49-68.

[47] D. Idczak, *On a generalization of a global implicit function theorem*, Advanced Nonlinear Studies 16 (1) (2016); DOI: 10.1515/ans-2015-5008.

[48] D. Idczak, *Functions of finite fractional variation and their applications to fractional impulsive equations*, Czechoslovak Mathematical Journal 67 (142) (2017), 171-195; DOI: 10.21136/CMJ.20170455-15.

[49] T. Kaczorek, D. Idczak, *Cauchy formula for the time-varying linear systems with Caputo derivative*, Fractional Calculus and Applied Analysis 20 (2) (2017), 494-505; DOI: 10.1515/fca-2017-0025.

[50] D. Idczak, R. Kamocki, *Existence of optimal solutions to Lagrange problem for a fractional nonlinear control system with Riemann-Liouville derivative*, Mathematical Control and Related Fields 7 (3) (2017), 449-464; DOI: 10.3934/mcrf.2017016.

1.2 Publications in proceedings - full texts

[51] D. Idczak, S. Walczak, *On the controllability of continuous Roesser systems*, Proceedings of the Second International Symposium on Methods and Models in Automation and Robotics, Międzyzdroje, Polska, 1995, 115-119.

[52] D. Idczak, *The maximum principle for a continuous 2-D Roesser model with a terminal condition*, Proceedings of the Third International Symposium on Methods and Models in Automation and Robotics, Międzyzdroje, Polska, 1996, 197-200.

[53] D. Idczak, *Nonlinear Roesser problem with a terminal condition. Maximum principle*, Proceedings Volume from the IFAC Workshop - Manufacturing Systems: Modelling, Management and Control, Vienna, Austria, 1997, 197- 202.

[54] D. Idczak, *The bang-bang principle for the Goursat-Darboux problem*, Electronic Proceedings of 15th International Symposium on the Mathematical Theory of Networks and Systems, University of Notre Dame, Indiana, USA (2002).

[55] D. Idczak, M. Majewski, *Nonlinear positive 2D systems and optimal control*, L. Benvenuti, A. De Santis and L. Farina (Eds.): Positive Systems - Proceedings of the First Multidisciplinary International Symposium on Positive Systems: Theory and Applications (POSTA 2003), Lecture Notes in Control and Information Sciences 294 (2003), 329-336.

[56] D. Idczak, S. Walczak, *Positive systems with nondecreasing controls. Existence and well-posedness*, L. Benvenuti, A. De Santis and L. Farina (Eds.): Positive Systems - Proceedings of the First Multidisciplinary International Symposium on Positive Systems: Theory and Applications (POSTA 2003), Lecture Notes in Control and Information Sciences 294 (2003), 369-376.

[57] D. Idczak, S. Walczak, *Optimal control of positive 2-D systems with infinite horizon*, Proceedings Sixteenth International Symposium on Mathematical Theory of Networks and Systems, Leuven, Belgia, 2004.

[58] D. Idczak, M. Majewski, *Controllability of Goursat-Darboux systems – some numerical results*, Preprints of the 16th IFAC World Congress, Praga, Czechy, 2005.

[59] D. Idczak, *On a continuous variant of a linear repetitive process*, Proceedings of the Fourth International Workshop on Multidimensional (nD) Systems NDS 2005, Wuppertal, Niemcy, 2005, 247-252.

[60] D. Idczak, S. Walczak, *Positive continuous Roesser systems*, Proceedings of 17th International Symposium on Mathematical Theory of Networks and Systems, Kyoto, Japan, 2006, 2451-2457.

[61] D. Idczak, *Bang-bang principle for a continuous repetitive process with distributed and boundary controls*, Proceedings of 17th International Symposium on Mathematical Theory of Networks and Systems, Kyoto, Japan, 2006, 803-809.

[62] D. Idczak, M. Majewski, S. Walczak, *On controllability of nonlinear systems described by ordinary differential equations*, C. Commault and N. Marchand (Eds.): Positive systems - Proceedings of the Second Multidisciplinary International Symposium on Positive Systems: Theory and Applications (POSTA 2006), Lecture Notes in Control and Information Sciences 341 (2006), 287-294.

[63] D. Idczak, R. Kamocki, *Some remarks on the point controllability over all passes for differential repetitive processes with control constraints*, Proceedings of the 2007 International Workshop on Multidimensional (nD) Systems, Aveiro, Portugal, 2007.

[64] D. Idczak, S. Walczak, *Linear-quadratic optimal control problem of second order with infinite time horizon*, Proceedings of the 2007 International Workshop on Multidimensional (nD) Systems, Aveiro, Portugal, 2007.

[65] D. Idczak, S. Walczak, *Compactness of fractional imbeddings*, Proceedings of the 17th International Conference on Methods and Models in Automation and Robotics, Międzyzdroje, Poland (2012), 585-588.

[66] D. Idczak, M. Majewski, *Compactness of fractional imbeddings for boundary value problems*, Proceedings of the 18th International Conference on Methods and Models in Automation and Robotics, Międzyzdroje, Poland (2013), 599-603.

[67] D. Idczak, R. Kamocki, M. Majewski, *Fractional continuous Roesser model with Riemann-Liouville derivative*, Proceedings of the 8th International Workshop on Multidimensional Systems (nDS'13), Erlangen, Germany, 2013.

1.3 Papers accepted for publication in scientific journals

[68] D. Idczak, R. Kamocki, M. Majewski, *On a fractional continuous counterpart of Fornasini-Marchesini model*, Journal of Integral Equations and Applications.

[69] D. Idczak, S. Walczak, *Necessary optimality conditions for an integro-differential Bolza problem via Dubovitskii-Miljutin method*, Discrete and Continuous Dynamical Systems, Series B.

1.4 Description of the obtained results

My research interests are mainly focused on the theory of differential equations, optimal control theory and controllability theory as well as selected issues of a basic nature in the field of nonlinear analysis.

1.4.1 Doctoral thesis

In doctoral thesis „*Optimization of systems described by partial differential equations*" control systems with distributed parameters of form (12) and (13) (see page 16), are investigated. For system (12) theorems on the existence, uniqueness and continuous dependence of solution (z^1, z^2) on control u have been obtained. Moreover, conditions guarantying the existence of a solution to optimal control problem described by linear control system and integral nonlinear cost functional have been given. Also, necessary optimality conditions for such a problem have been derived. For system (13), a theorem on the existence and uniqueness of a solution z for any fixed control u has been proved. Necessary optimality conditions for optimal control problem described by semilinear control system, pointwise terminal conditions and integral nonlinear cost functional have been obtained, too.

Results of the doctoral thesis or some generalizations of them have been published and will be commented in the remaining part of the report.

1.4.2 Habilitation dissertation

In habilitation dissertation *"On some variational problems for ordinary differential equations with Dirichlet and periodic boundary conditions and their application",* the ordinary control systems of second and higher order are investigated with the aid of variational methods. Presented approach and results obtained for control systems have been applied to study optimal control problems described by such systems. Dissertation is based on works [14], [15], [18] ([1]) and consists of three parts.

[1] number markings of bibliographic entries relate to the author's and co-author's papers and are consistent with the list of publications placed at the beginning of the report; in the case of works of other authors, letter markings, or letter-number markings are used, and are consistent with the list of these works placed at the end of chapter 1.4.7 of the report

In part 1 „*Stability of semilinear Dirichlet problems*", the results of paper [18] are contained (results of the paper [18] as well as papers [14] and [15] will be discussed in the remaining part of the report). One of them has been used in the proof of an additional theorem on the existence of a solution to an optimal control problem described by control system (1) and integral nonlinear cost functional.

In part 2 „*Matrix-periodic problem of order 2k*", the results obtained in paper [14] are contained. Moreover, applications of these results to study the continuous dependence of solutions to control system on controls and, similarly as in Part 1, in the proof of an additional existence theorem for an optimal control problem described by such a system and an integral nonlinear cost functional, are given.

In part 3 „*Coercive Dirichlet problem of order 2k and its optimization*", the results obtained in paper [15], are contained. Also in this part of the dissertation, next to the results obtained in [15], a theorem on the existence of a solution to an appropriate optimal control problem is obtained.

1.4.3 Works on differential equations

Papers discussed in this part of the report concern initial value and boundary value problems for ordinary and partial differential equations as well as integro-differential equations, with derivatives of integer and noninteger orders. In the „noninteger" case, the derivatives in Riemann-Liouville sense and Caputo sense were considered.

Ordinary equations

In paper [18], we derived the theorems on continuous dependence of solutions functional parameters (controls) ([2]) for a semilinear system of the second order, with Dirichlet boundary conditions

$$\begin{cases} \ddot{u} + au = D_u F(t, u, \omega), \ t \in I := [0, \pi] \text{ a.e.}, \\ u(0) = u(\pi) = 0 \end{cases} \tag{1}$$

where $F : I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $D_u F$ denotes the gradient of F in u, without assumptions guarantying the uniqueness of a solution. The question of continuous dependence of solutions on controls for ordinary Dirichlet problem of the second order, in the case of $n = 1$ and under conditions guarantying the uniqueness of a solution was investigated among others in [Ga], [Kla], [Ing], [LP], [Se1], [VK], [BL]. In [18], first, using the dual least action principle ([Maw]), we derive theorems on continuous dependence of solutions on right side for semilinear operator equation

$$Lu = \nabla g(u), \tag{2}$$

[2] the continuous dependence of the solutions on the functional parameters or, more generally, on the right side of the system is sometimes referred to as system stability; one should distinguish between the concept of system stability and the concept of stability of the solution of the system referred to in the following part of the report

where $L : D(L) \subset H \to H$ is linear self-adjoint operator with closed range, $g : H \to \mathbb{R}$ - convex and continuous function on H, Gateaux differentiable on the domain of the operator L, H - real Hilbert space. Next, we used the results obtained for the above operator equation to obtain theorems on the continuous dependence of solutions on functional parameter $\omega : I \to \mathbb{R}^m$ for Dirichlet problem (1). We consider the case of $a \geq 1$, i.e. when the functional of action for the above equation is not coercive ([3]). Coercive case ($a < 1$), also without assumptions guarantying the uniqueness of a solution was considered in paper [Wal3] with the aid of the direct method of calculus of variations.

In paper [14], using the direct method of calculus of variations, we investigate some nonlinear problem of even order ($2k$) with the matrix-periodic boundary conditions described with the aid of a matrix A. Due to the complicated notation of this problem, below we will provide only some of its specific cases.

If $k = 2$, problem takes the form (for the shortening of the natation, we omit the symbol of the variable t as an argument of the function u and its derivatives)

$$\frac{d}{dt}(\frac{d}{dt}\ddot{u} - D_{u_1}G(t, u, \dot{u})) + D_uG(t, u, \dot{u}) = 0, \ t \in I \text{ a.e.}, \tag{3}$$

$$\begin{bmatrix} u(0) \\ \dot{u}(0) \end{bmatrix} = A \begin{bmatrix} u(\pi) \\ \dot{u}(\pi) \end{bmatrix},$$

$$\begin{bmatrix} \ddot{u} \mid_{t=0} \\ (\frac{d}{dt}\ddot{u} - D_{u_1}G(t, u, \dot{u})) \mid_{t=0} \end{bmatrix} = B \begin{bmatrix} \ddot{u} \mid_{t=\pi} \\ (\frac{d}{dt}\ddot{u} - D_{u_1}G(t, u, \dot{u})) \mid_{t=\pi} \end{bmatrix}$$

where $G = G(t, u, u_1) : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $A = \begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{bmatrix}$ is a nonsingular matrix such that $A^T = A^{-1}$ and

$$B = \begin{bmatrix} a_{1,1} & -a_{0,1} \\ -a_{1,0} & a_{0,0} \end{bmatrix}.$$

If the function G is sufficiently smooth, equation (3) takes the normal form

$$u^{(4)} + H(t, u, \dot{u}, \ddot{u}) = 0, \ t \in I \text{ a.e.},$$

where

$$\begin{aligned} H(t, u, u_1, u_2) = -D_{u_1 t}G(t, u, u_1) - D_{u_1 u}G(t, u, u_1)u_1 \\ - D_{u_1 u_1}G(t, u, u_1)u_2 + D_uG(t, u, u_1). \end{aligned}$$

In the case of any k, $A = I$ and a function G not depending on $u_1, ..., u_{k-1}$ ($G = G(t, u)$), the problem reduces to a periodic problem

$$\begin{cases} u^{(2k)} + (-1)^k D_uG(t, u) = 0, \ t \in I \text{ a.e.} \\ u^{(i)}(0) = u^{(i)}(\pi), \ i = 0, ..., 2k - 1 \end{cases}.$$

[3] by coercivity of a functional $f : W \to \mathbb{R}$ where W is a normed space, we mean the convergence $f(u) \underset{\|u\| \to \infty}{\longrightarrow} \infty$

When $A=-I$ and $G=G(t,u)$, we obtain an antiperiodic problem

$$\begin{cases} u^{(2k)}+(-1)^k D_u G(t,u)=0,\ t\in I \text{ a.e.} \\ u^{(i)}(0)=-u^{(i)}(\pi),\ i=0,...,2k-1 \end{cases}.$$

In monograph [MW], using the direct method of calculus of variations and Du Bois-Reymond lemma, the authors prove a theorem on the existence of a solution to a periodic problem of second order of type

$$\begin{cases} \ddot{u}=D_u G(t,u),\ t\in I \text{ a.e.,} \\ u(0)=u(\pi),\ \dot{u}(0)=\dot{u}(\pi) \end{cases}.$$

The basic assumption is the so-called „coercivity on the kernel”([4])

$$\int_I G(t,c)dt \underset{|c|\to\infty}{\longrightarrow} \infty. \tag{4}$$

In paper [14], we obtained an analogous theorem with an appropriate generalization of condition (4), for problem of order $2k$ with matrix-periodic boundary conditions. This condition is of the form

$$\int_I F(t,W(t),W'(t),...,W^{(k-1)}(t))dt \underset{\sum_{i=0}^{k-1}|c_i|\to\infty}{\longrightarrow} \infty$$

where $W(t)=c_0+c_1t+...+c_{k-1}t^{k-1}$, F is a function describing the equation. The proof of this theorem is based on a Du Bois-Reymond lemma for functions of one variable and derivatives of order k satisfying the matrix-periodic boundary conditions, derived in [14]. Periodic problems of higher order were investigated by other authors with the aid of topological methods. In paper [MZ], the authors study an equation of fourth order ($k=2$). The method used in [MZ] can not be applied in the case of $k>2$, as the authors mention in their work. In [OZ], the problems of odd order are investigated.

In paper [15], using the direct method of calculus of variation, we obtained the existence of a solution to Dirichlet problem of the second order of the form

$$\begin{cases} \frac{d}{dt}D_{\dot{u}}F(t,u,\dot{u},\omega)=D_uF(t,u,\dot{u},\omega),\ t\in I \text{ a.e.} \\ u(0)=u(\pi)=0 \end{cases} \tag{5}$$

where $F: I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$.

In paper [28], the following second order system

$$\ddot{u}=D_uF(t,u,\omega),\ t\in[0,\infty) \text{ a.e.,} \tag{6}$$

with initial condition

$$u(0)=\alpha$$

[4]for the first time this condition was considered in [ALP]

is considered (systems on unbounded interval $[0,\infty)$ are called the infinite horizon systems). A new concept of stability of zero solution for second-order systems has been proposed here. More precisely, the notion of the asymptotic stability of the zero solution (with respect to the initial value α and the functional parameter ω) in the variational sense has been introduced and conditions ensuring such stability have been given. The problem of stability of the solutions for second order systems was investigated by a few authors. In [Le], [Se2], the stability of solutions of the second order semilinear systems is tested using the Lapunov function. In our paper, this type of function is not used. It is worth noting that in the adopted definition of variational stability we do not impose conditions on the value of the first derivative at the initial moment, whereas the application of the method consisting in replacing the second order system with the appropriate first order system and using the classical stability results for first order systems would lead to a concept of stability in which such conditions would occur. The main result of the work is illustrated by an example of a system whose zero solution is not stable in the classical sense but is stable in a variational sense.

Partial equations In the work [12], a generalization of the Du Bois-Reymond lemma for functions of two variables satisfying boundary conditions of Dirichlet type, and mixed second order partial derivatives ([Wal2]) to the case of functions of two variables and mixed partial derivatives of any order, is proved. Next, it is proved that system

$$\frac{\partial^{2k+2l} z}{\partial x^{2k} \partial y^{2l}} = D_z H(x,y,z),\ (x,y) \in P := [0,1] \times [0,1] \subset \mathbb{R}^2 \text{ a.e.},$$

with Dirichlet boundary conditions has a weak solution satisfying some additional regularity condition. Proof is based on the direct method of the calculus of variations.

In paper [9], a Du Bois-Reymond lemma for functions of two variables satisfying Dirichlet boundary conditions and for partial derivatives of the first order has been deduced. This lemma served to prove, with the aid of the direct method of calculus of variations, the existence of a unique solution to a partial integro-differential problem.

Ordinary integro-differential equations In paper [40], we derived existence of a unique solution to a Cauchy problem for an integro-differential equation of Fredholm type

$$u'(t) = \int_a^b F(t,\tau,u(\tau))d\tau + \alpha(\tau),\ t \in [a,b] \text{ a.e.}$$

with initial condition

$$u(a) = 0,$$

in the space of absolutely continuous functions with square integrable derivatives. We also obtained a sensitivity theorem, i.e. a theorem on the continuous differentiability of a mapping which to any functional parameter $\alpha \in L^2$ assigns the corresponding solution to the above problem. These results have been obtained with the aid of a theorem on the diffeomorphism derived in paper [35] (see part 1.4.6).

Fractional equations Another trend of my research is the systems of differential equations (including repetitive processes and integro-differential equations), containing fractional derivatives in the sense of Riemann-Liouville and Caputo. Fractional differential calculus and its applications in the theory of differential equations are currently undergoing intensive development. Although there is currently no physical theory that leads to equations of state containing fractional derivatives, it turns out that many anomalous processes are better modeled by systems containing fractional derivatives, rather than by systems containing only derivatives of natural orders. It should be mentioned here diffusion processes in porous media ([La], [ACV]), viscoelastic processes for plasma substances ([BT], an electric supercapacitor model ([WE]).

In paper [33], we obtained a theorem on the existence and uniqueness of a solution to nonlinear problem

$$\begin{cases} D^{\alpha}_{a+}x(t) = f(t, x(t)), \ t \in [a, b] \text{ a.e.} \\ I^{1-\alpha}_{a+}x(a) = c \end{cases} \tag{7}$$

where $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$, $c \in \mathbb{R}^n$ and $D^{\alpha}_{a+}x$, $I^{\alpha}_{a+}x$ are (left-sided) Riemann-Liouville derivative and integral of order $\alpha \in (0, 1)$ of a function $x(\cdot)$. Moreover, we derived the Cauchy formula for the solution of an autonomous linear system

$$D^{\alpha}_{a+}x(t) = Ax(t) + b(t).$$

In the limit case of $\alpha = 1$, this formula reduces to the classical one for the solution of an ordinary linear system of the first order. In [49], we obtained analogous results for problem

$$\begin{cases} {}^{C}D^{\alpha}_{a+}x(t) = f(t, x(t)), \ t \in [a, b] \text{ a.e.} \\ x(a) = c \end{cases}$$

and for linear nonautonomous problem

$$\begin{cases} {}^{C}D^{\alpha}_{a+}x(t) = A(t)x(t) + b(t), \ t \in [a, b] \text{ a.e.} \\ x(a) = c \end{cases}$$

where ${}^{C}D^{\alpha}_{a+}x$ denotes the Caputo derivative of a function $x(\cdot)$.

In [66], using the direct method of the calculus of variations, a theorem on the existence of a solution to a certain potential equation containing fractional derivatives of two different orders has been proved. Previously, problems of this type were investigated, but for systems containing only one fractional derivative ([Bo], [43]). The analysis of the general problem was made possible by the use of a theorem on the compactness of the imbedding for the spaces of functions possessing Riemann-Liouville derivatives, obtained in [65] (see also theorems on the compactness of the imbedding for Sobolev spaces of fractional order introduced and studied in [39]).

Paper [38] concerns the repetitive processes of fractional order

$$\begin{cases} D^{\alpha}_{a+}z_{k+1}(t) = A_1 z_{k+1}(t) + A_2 w_k(t) + B u_{k+1}(t) \\ w_{k+1}(t) = C_1 z_{k+1}(t) + C_2 w_k(t) + D u_{k+1}(t) \end{cases} \tag{8}$$

where $z_k(t) \in \mathbb{R}^n$, $w_k(t) \in \mathbb{R}^m$, $u_k(t) \in \mathbb{R}^r$ for $k \in \mathbb{N} \cup \{0\}$, coefficients on the right side of the system are matrices of appropriate dimensions, $D^{\alpha}_{a+}z_{k+1}$ means the derivative of order $\alpha \in (0,1)$ in the sense of Riemann-Liouville or Caputo. The above process is a fractional counterpart of the discrete-continuous repetitive process known in the theory of automatic control (also named the differential repetitive process), which is used, among others, to model the coal mining and the metal rolling processes. In the first case the process is considered with initial conditions

$$\begin{cases} (I^{1-\alpha}_{a+}z_k)(a) = c_k \text{ for } k \in \mathbb{N}, \\ w_0(t) = f(t) \text{ for } t \in [a,b], \end{cases},$$

while in the second case - with conditions

$$\begin{cases} z_k(a) = c_k \text{ for } k \in \mathbb{N}, \\ w_0(t) = f(t) \text{ for } t \in [a,b], \end{cases}.$$

where $c_k \in \mathbb{R}^n$ for $k \in \mathbb{N}$ and $f : [a,b] \to \mathbb{R}^m$ are initial data. In both cases, we obtained theorems on the existence and uniqueness of a solution

$$\mathfrak{z} \in \{\mathfrak{z} = (z_k(\cdot))_{k\in\mathbb{N}} : [a,b] \to \prod_{k=1}^{\infty} \mathbb{R}^n;\ z_k(\cdot) \in AC^{\alpha}_{a+}([a,b],\mathbb{R}^n),\ k \in \mathbb{N}\}$$

for any fixed control $\mathfrak{u} = (u_k(\cdot))_{k\in\mathbb{N}} : [a,b] \to \prod_{k=1}^{\infty} \mathbb{R}^r$ belonging to an appropriate class (here $AC^{\alpha}_{a+}([a,b],\mathbb{R}^n)$ denotes the set of functions possessing Riemann-Liouville, Caputo derivatives of order α, respectively), as well as theorems on the continuous dependence of solutions on controls. In the case of Caputo derivative, we also proved a theorem on the approximative controllability (this result is discussed in the part 1.4.5).

In paper [37], we deduced a theorem on the existence and uniqueness of a solution $x(\cdot)$ as well as on the differentiable dependence of this solution on functional parameter $u(\cdot)$ for integro-differential Cauchy problem of Volterra type

$$\begin{cases} D^{\alpha}_{a+}x(t) = \int_a^t \Phi(t,s,x(s))ds + u(t),\ t \in [a,b] \text{ a.e.} \\ I^{1-\alpha}_{a+}x(a) = 0 \end{cases} \tag{9}$$

Similarly, as in paper [40], proof of this result is based on the theorem on diffeomorphism from [35].

In paper [45], we investigate analogous issues as in [37], for problem

$$\begin{cases} D^{\alpha}_{a+}x(t) + \int_a^t \Phi(t,s,x(s),u(s))ds = f(t,x(s),v(s)),\ t \in [a,b] \text{ a.e.} \\ I^{1-\alpha}_{a+}x(a) = 0 \end{cases}$$

with two functional parameters $u(\cdot)$, $v(\cdot)$, involved nonlinearly (in the case of system (9) control $u(\cdot)$ acts on the system in the linear way). Because of this nonlinearity, one can not use the theorem on diffeomorphism. One can, however, apply a theorem on the global implicit function obtained in the work [41], which is a generalization to the nonlinear operators (with respect to the parameter) of the diffeomorphism theorem (see part 1.4.6).

Next two papers, [67] and [68], concern partial differential equations of fractional order with Riemann-Liouville derivatives. In [67], a fractional variant of the system (12) is studied, namely

$$\begin{cases} D^{\alpha}_{a+,x}z^1 = f^1(x,y,z^1,z^2,u) \\ D^{\beta}_{c+,y}z^2 = f^2(x,y,z^1,z^2,u) \end{cases} \tag{10}$$

a.e. on $P = [a,b] \times [c,d] \subset \mathbb{R}^2$, with initial conditions

$$\begin{cases} (I^{1-\alpha}_{a+,x}z^1)(a,y) = 0,\ y \in [c,d] \text{ a.e.} \\ (I^{1-\beta}_{c+,y}z^2)(x,c) = 0,\ x \in [a,b] \text{ a.e.} \end{cases} \tag{11}$$

where $(\alpha,\beta) \in (0,1]\times(0,1]$, $f^1 : P\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^m \to \mathbb{R}^{n_1}$, $f^2 : P\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^m \to \mathbb{R}^{n_2}$, (here $D^{\alpha}_{a+,x}$, $D^{\beta}_{c+,y}$ and $I^{1-\alpha}_{a+,x}$, $I^{1-\beta}_{c+,y}$ mean the Riemann-Liouville derivatives and integrals of fractional order with respect to the appropriate variables). The main result of the work, next to the theorem on the integral representation of a function having the appropriate fractional derivative, is a theorem on the existence and uniqueness of a solution $(z^1(\cdot), z^2(\cdot))$ to the above problem and on the continuous dependence of the solution on the control $u(\cdot)$. In [68], we study a fractional variant of system (13), i.e.

$$D^{\alpha,\beta}_{a+,x,c+,y}z(x,y) = f(x,y,z(x,y),D^{\alpha}_{a+,x}z(x,y),D^{\beta}_{c+,y}z(x,y),u(x,y)),\ (x,y) \in P \text{ a.e.},$$

with initial conditions

$$\begin{cases} I^{1-\alpha,1-\beta}_{a+,x,c+,y}z(x,c) = \gamma(x),\ x \in [a,b] \\ I^{1-\alpha,1-\beta}_{a+,x,c+y}z(a,y) = \delta(y),\ y \in [c,d] \end{cases}$$

where $\gamma : [a,b] \to \mathbb{R}^n$, $\delta : [c,d] \to \mathbb{R}^n$ are absolutely continuous functions satisfying condition $\gamma(a) = \delta(c)$, while $D^{\alpha,\beta}_{a+,x,c+,y}z$, $D^{\alpha}_{a+,x}z$, $D^{\beta}_{x,c+,y}z$ and $I^{1-\alpha,1-\beta}_{a+,x,c+,y}z$ are the appropriate Riemann-Liouville partial derivatives and fractional integral of a function $z(\cdot)$. The concept of a mixed partial derivative $D^{\alpha,\beta}_{a+,x,c+,y}z$ of order (α,β) has been introduced in the work and a theorem on the integral representation of functions having such derivative has been derived. Also in this paper, a theorem on the existence and uniqueness of a solution $z(\cdot)$ to the above problem as well as on the continuous dependence of it on control $u(\cdot)$ has been proved. An existence theorem for the above problem, assuming that the right side of the system does not depend on the single derivatives and with a slightly different definition of a mixed fractional derivative, can be found in [VG].

1.4.4 Works on optimal control theory

I dealt with the theory of optimal control in the context of the issue of the existence of optimal solutions and necessary conditions of the first order for optimality. The main subject of my interests were the following control systems with distributed parameters:

$$\begin{cases} \frac{\partial z^1}{\partial x} = f^1(x, y, z^1, z^2, u) \\ \frac{\partial z^2}{\partial y} = f^2(x, y, z^1, z^2, u) \end{cases} \tag{12}$$

for $(x, y) \in P$ a.e., with initial conditions

$$z^1(0, y) = 0,\ z^2(x, 0) = 0$$

for $x, y \in [0, 1]$ and

$$\frac{\partial^2 z}{\partial x \partial y} = f(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, u). \tag{13}$$

for $(x, y) \in P$ a.e., with initial conditions

$$z(x, 0) = 0,\ z(0, y) = 0$$

for $x, y \in [0, 1]$ (in some papers, the above systems are considered with nonzero initial conditions)([5]). In addition, I studied control systems with distributed parameters (14) (see page 20), as well as ordinary second order control systems with infinite horizon of the form (6) and (15) (see page 22).

Partial systems Systems of type (12) are used to describe chemical processes occurring in reactors with varying catalyst activity. The natural space of solutions (in the Caratheodory sense) for these systems is the product of the spaces of functions that are absolutely continuous with respect to the appropriate variables. Such functions can also be defined (with precision „a.e.") by means of generalized derivatives and treated as elements of the appropriate Sobolev spaces.

In paper [10], we derived theorems on the existence of a unique solution and on the continuous dependence of solutions on controls for the system (12) as well as on the existence of an optimal solution for a linear nonautonomous system of type (12) considered with an integral nonlinear cost functional, assuming convexity of the

[5] system (13) is called the Goursata-Darboux system; in the literature concerning automatic control, discrete counterparts of (12) and (13) are known as Roesser systems and Fornasini-Marchesini system

integrand with respect to the set of state and control variables. Work [13] contains the proof of the maximum principle for the optimal control problem described by the system (12) and integral nonlinear cost functional. This principle has a pointwise character, and the conjugate system is a linear system of type (12).

In paper [MS], a theorem on the existence of a unique solution and on the continuous dependence of solutions on controls for the system (12) with initial conditions understood in the integral sense, studied with the aid of the notions of a generalized derivative and trace of a function, has been obtained. It is worth mentioning that in the work [MS], in the proof of the existence theorem one uses the theorem on contracting k-fold iteration of a mapping, while in the work [10] we use the concept of Bielecki metric and Banach contraction principle. The method based on Bielecki metric allows to avoid the complicated calculations leading to some estimates for k-fold iteration of the mapping under consideration. The theorem on the continuous dependence of solutions on controls obtained in [MS] has stronger assumptions and stronger thesis than the theorem obtained in [10]. In the work [VS], a maximum principle is obtained for a problem connected with a cost functional depending only on the solution and considered in the class of piecewise continuous controls and in a special class of solutions.

In paper [44], we consider, among others, a special case of system (12), namely

$$\begin{cases} \frac{\partial z^1}{\partial x} = h^1(x,y,z^2) + B^1(x,y)u^1 \\ \frac{\partial z^2}{\partial y} = h^2(x,y,z^1) + B^2(x,y)u^2 \end{cases}$$

with zero initial conditions and cost functional

$$J(u) = \int_P f^0(x,y,z_u^1(x,y),z_u^2(x,y),u^1(x,y),u^2(x,y))dxdy \to \min.,$$

where $(z_u^1(\cdot), z_u^2(\cdot))$ is a solution to the system, corresponding to control $(u^1(\cdot), u^2(\cdot))$. This problem is considered in nonstandard spaces of solutions (absolutely continuous functions of two variables) and controls (functions of two variables, absolutely continuous with respect to the appropriate variables). Assuming convexity of f^0 in (u^1, u^2) we proved a theorem on the existence of a solution to the above problem.

The optimal control problem described by the system (12) and an integral nonlinear cost functional, under pointwise „end" conditions imposed on the solution, is studied in papers [52], [53] and [19]. Consideration of the pointwise „end" conditions means that we have to study the system in the spaces of solutions and controls such as in [44]. In [52], necessary optimality conditions have been obtained for a linear autonomous system of type (12). The assumption of linearity and autonomy made it possible to bring the tested problem to a system of type (13) ([6]). In paper [53], the system (12) nonlinearly depending on the state variable is examined. Consequently, it can not be reduced to the system (13). In this case, the necessary

[6]in [Pie], the equivalence of both systems is shown in the linear autonomous case

optimality conditions have been derived using the extremum principle for a smooth-convex problem. In paper [19], in the linear nonautonomous case, a theorem on the continuous dependence of solutions on controls and a theorem on the existence of an optimal solution are derived.

Systems of type (13), applicable to describe the process of gas absorption and drying ([TS]), were investigated by many authors. In the work [AO], for a system without a control variable, the existence of a classical solution (i.e. having continuous corresponding partial derivatives) is proved and the continuous dependence of solutions on initial conditions and the right side of the equation is examined. The natural space of solutions for systems containing control is the space of functions having appropriate integral representation (solutions in the Caratheodory sense). Absolutely continuous functions of two variables defined in [Wal1] with the aid of an absolutely continuous function of an interval([7]) possess such a representation. In papers [Sur1], [Sur2], [Sur3], the solution space is defined as a Sobolev type space, described using generalized derivatives. It can be shown that the elements of such a space (with an accuracy to the representation a.e.) have an appropriate integral representation.

In paper [PS1], the authors obtained a theorem on the existence and uniqueness of a local solution to system (13), while in [Sur3] (see also [Sur1]) existence of a unique global solution in the Caratheodory sense is proved. The proof of this theorem is based on the theorem on the contracting k-fold iteration of a mapping. In [7], we obtained existence of a unique global solution using Bielecki metric and Banach contraction principle. This allowed, as in the case of the system (12), to simplify the proof considerably.

The main result of the work [7] is a theorem on the existence of optimal solutions for the system (13) considered in the class of controls of two variables of finite variation on P ([6]), with an integral nonlinear cost functional. This theorem is obtained on the basis of derived in [7] the Helly's choice principle for functions of two variables with finite variation, analogous to the Helly's principle for functions of one variable. This principle met with the interest of specialists in the field of the theory of real functions ([BBCh], [Ch], [ChT1], [ChT2], [ChT3]). In [5], a case of a functional dependent also on the value of variation of control u on the set P is considered.

In paper [8], we obtained a theorem on the existence of optimal solutions for nonlinear system (13) considered with cost functional containing nonlinear integral term and terms depending on the rapidity of the changes of a control and on the

[7]Independently, the concept of an absolutely continuous function of two variables was also introduced in the work [BG], but the authors do not use the notion of the function of an interval; some authors with absolutely continuous functions of two variables explicitly call functions that have an appropriate integer representation (see e.g. [PS1]). A quality of the definition given in [Wal1] is a hint to define in an analogous way, i.e. using the concept of the function of an interval associated with the function of two variables (similarly, as in the case of functions of one variable) a function of two variables of finite variation ([6]).

number of switching of it. This theorem is an extension of the existence theorem proved in paper [3] for linear system of type (13) and cost functional containing integral linear term and depending on the rapidity of the changes of a control. In [3], apart from the existence of optimal solutions, the maximum principle for the problem under consideration is also derived.

In [11], it is proved that there are optimal solutions for the problem described by the linear nonautonomous system (13) and the nonlinear cost functional, which is the sum of three integral components with integrands that depend in a convex way, in addition to the control and state variables, also on the variables representing the partial derivatives of the solution. This problem is considered in the class of summable controls, taking their values in a fixed convex set. In the same control class, in paper [17], a linear nonautonomous system (13) is considered together with the integral cost functional with the integrand being convex only with respect to the control variable. Here, the concept of the family of functions of two variables equiabsolutely continuous is introduced and next, a characterization of such a family is given and Arzela-Ascoli theorem for absolutely continuous functions of two variables is proved. It is also proved that from any sequence of solutions corresponding to the controls of the considered class, it is possible to choose a subsequence uniformly convergent to a solution corresponding to certain control and such that the sequences of corresponding partial derivatives of these solutions converge weakly in the space of the summable functions to the derivatives of the limit solution. Hence, a theorem on the existence of an optimal solution for the problem under examination is deduced.

The existence of a solution to an optimal control problem for a more general system, but considered with the Mayer-type cost functional, depending only on the value of the solution at a fixed „end" point, is obtained in [Sur2], assuming the convexity of the so-called set of velocities. In particular, this means that one can not compare the result obtained in [17] and the result obtained by replacing the problem with the integral cost functional by the problem with the cost functional depending only on the value of the solution at the „end" point and applying the result obtained in work [Sur2].

In paper [55], positive systems of type (13) are studied, i.e. systems whose solutions are nonnegative provided that the controls are nonnegative. In the first part of the paper, the conditions guaranteeing the „positivity" of the nonlinear system are given. In the second part, we prove the existence of an optimal solution for the problem described by the positive system, linear with respect to first order partial derivatives, and the integral nonlinear cost functional, assuming the convexity of the so-called set of generalized velocities of type (16) (see page 23). It should be noted that in the obtained theorem one does not impose the Lipschitz condition on the integrand in the cost functional, while the application of the method consisting in replacing the problem with the integral cost functional by the problem with the cost functional depending only on the value of the solution at the „end" point and using the result obtained in [Sur2], leads to such an assumption. It is worth adding that the proof presented in [55] is direct, while in [Sur2] an advanced tool (lower closure theorem) is used.

In paper [4], a maximum principle is derived for the problem described by n-dimensional system of type (13) (nonlinear only in state variable), pointwise „end" conditions imposed on the solution and integral nonlinear cost functional. In the general case, the conjugate system is given in the form of the integral equation, while with some additional assumptions it takes the form of a linear system of type (13). The necessary conditions for the two-dimensional problem described by the system (13) and the cost functional depending only on the value of the solution at the „end" point, without pointwise constraints, are obtained in [Sur1], [PS2], [Sr] (in [Sr], the class of controls piecewise continuous with respect to each variable is considered).

In paper [23], the continuous dependence of the set of solutions to optimal control problem on the data describing the problem for systems of type (13) (linear and nonlinear), is examined. This dependence is described with the aid of the notion of the upper limit of the sequence of sets.

In paper [57], the system (13) is considered on the unbounded interval of type $[0,\infty)\times[0,\infty)$, in the class of locally absolutely continuous solutions and locally summable controls. The inspiration for undertaking research on such systems was optimal control theory for ordinary systems with an infinite horizon, to which monograph [CH] is devoted. In the first part of the work, a theorem on the existence and uniqueness of a solution is obtained and the question of the „positivity" of the system is examined. In the second part of the work, various concepts of optimality of solutions to the system under consideration are introduced, related to the cost functionals of integral type. Relations between these concepts are indicated and the so-called principle of optimality is deduced. This principle says, generally speaking, that some of the types of the tested concepts of optimality (on the interval $[0,\infty)\times[0,\infty)$) imply „finite" optimality, i.e. optimality on every finite subinterval $[0,X]\times[0,Y]$ of the interval $[0,\infty)\times[0,\infty)$.

In papers [59], [26], [32], we study a system of partial differential equations of type

$$\begin{cases} \frac{\partial^2 z}{\partial t\partial x} = A_0 z + A_1\frac{\partial z}{\partial t} + A_2\frac{\partial z}{\partial x} + B_0 w + k \\ \frac{\partial w}{\partial t} = D_0 w + E_0 z + E_1\frac{\partial z}{\partial t} + l \end{cases} \tag{14}$$

for $(t,x)\in[0,\infty)\times[0,1]$ a.e., with initial conditions

$$z(t,0)=\varphi(t) \text{ for } t\in[0,\infty),$$
$$w(0,x)=\psi(x) \text{ for } x\in[0,1],$$

where $z,k\in\mathbb{R}^n$, $w,l\in\mathbb{R}^m$. The above system is a continuous counterpart of both discrete and discrete-continuous repetitive processes.

Systems of the form (14) have not been previously studied by other authors. In [59], a theorem on the existence and uniqueness of a solution (z,w) corresponding to the control (k,l) and a theorem on the continuous dependence of solutions on controls, are obtained. In the work [26], the maximum principle for the problem described by the linear and autonomous process and the cost functional which depends on the „end" functions, is derived. The obtained result can be used to test the „functional" controllability of repetitive processes. In [32], the existence of an optimal solution is obtained for the problem described by the process (14) (in [32], the process is called a two-directionaly continuous repetitive process) and an integral nonlinear cost functional. From the obtained theorem, the existence of a solution to the problem considered in paper [26] results.

Ordinary systems The results on the existence of a solution to a second order control system, obtained in [15], have been applied in this work to derive, using the extremum principle for a smooth-convex problem ([IT]), the maximum principle for optimal control problem described by the system (5) with Dirichlet boundary conditions and integral nonlinear cost functional of the form

$$J(u,\omega) = \int_I f^0(t, u(t), \dot{u}(t), \omega(t))dt.$$

Maximum principle for scalar ($n = 1$) problem of the form

$$\frac{d}{dt}a(t,u,\dot{u}) = b(t,u,\dot{u},\omega),\ t \in I \text{ a.e.},$$
$$u(0) = u(\pi) = 0,$$

with integral nonlinear cost functional and under additional equality and inequality constraints, has been obtained in paper [GR] using the method of McShane variations.

Maximum principle from [15] has been used in paper [30] to derive the maximum principle for an optimal control problem connected with the second order control system with infinite horizon. In [30], besides the classical optimality, other types of optimality are considered, namely strong optimality, catching-up optimality, sporadically catching-up optimality and finite optimality. The principle of optimality is proved. It states that each of the above types of optimality, including classical one, implies finite optimality meaning optimality on every finite interval $[0,T]$. The main results of this work are two maximum principles, obtained with the aid of the results from [15]. The first principle concerns the general case of the problem under consideration and the second one - a special case. The second theorem is analogous to the appropriate maximum principle for optimal control problems with the first order systems and infinite horizon ([CH]).

In papers [31], [64], the optimal control problems described by ordinary second order systems with infinite horizon are studied, too.

In paper [31], existence of an optimal solution for the optimal control problem described by the system (6) is obtained in the case when the function F is linear with respect to ω and in the general case. Both problems are considered with an integral nonlinear cost functional.

In paper [64], we examine an optimal control problem described by control system of the form

$$\ddot{x} = A(t)x + B(t)u + v(t), \; t \in [0, \infty) \text{ a.e.}, \tag{15}$$

and integral quadratic cost functional. Theorem on the existence, uniqueness and continuous dependence of solutions on controls for the above equation as well as a theorem on the existence of an optimal solution have been obtained.

In the work [56], the optimal control problem described by the ordinary first order positive system (on a finite interval) and the Bolza type functional (the sum of the integral component and the pointwise component) is investigated. Theorem on the existence of optimal solutions and their continuous dependence on the data appearing in the description of the problem has been obtained here with the aid of the concept of Γ-convergence of functional sequences and the upper limit of the sequence of sets.

In the work [69], a maximum principle has been obtained for the Bolza problem associated with an ordinary integro-differential equation. The proof is based on the Dubovitski-Miljutin method, which made it possible to avoid the assumptions on convexity of functions occurring in the equation and in the cost function, with respect to the control variable. In the alternative approach, based on the smooth-convex principle due to Ioffe-Tikchomirov, some assumptions of the convexity type occur. It should be added that the minimum condition obtained in the work [69] contains the gradients of functions describing the equation and cost functional, while the minimum condition obtained by the extremum principle is expressed by the functions themselves.

Fractional systems In papers [42], [46], we investigate the linear-quadratic problems for ordinary control systems of fractional order with Riemann-Liouville and Caputo derivatives, respectively. In the case of the system with the Caputo derivative, the cost function, in addition to the point component also contains functional components. The main results of these works are formulas for gradients of tested cost functionals under given constraints and their application to proofs of maximum principles for the problems under considerations.

In paper [50], we study an ordinary nonlinear control system containing Riemann-Liouville derivative

$$\begin{cases} D^{\alpha}_{a+}x(t) = f(t, x(t), u(t)), \; t \in [a, b] \text{ a.e.} \\ I^{1-\alpha}_{a+}x(a) = c \end{cases},$$

with integral cost functional

$$\int_a^b f_0(t, x(t), u(t))ds \to \min .$$

and under control constraints of type

$$u(t) \in M \subset \mathbb{R}^m.$$

Existence of solutions to the above problem has been obtained in papers [Kam] and [PAT]. In [Kam], one assumes linear structure of the control system and convexity of f_0 in u, while in [PAT] one assumes that f_0 is convex in (x, u) and $M = \mathbb{R}^m$. In work [50], using a „fractional" Gronwall lemma proved in ([YGD]) and the theorem on the implicit function for multivalued mapping (see for example [Kis]) we derived a theorem on the existence of optimal solution under assumption on convexity of the set of generalized velocities

$$\{(v_0, v) \in \mathbb{R} \times \mathbb{R}^n;\ v_0 = f_0(t, x, u),\ v = f(t, x, u) \text{ for some } u \in M\} \tag{16}$$

for $t \in [a, b]$ a.e. and $x \in \mathbb{R}^n$.

In [44], the question of the existence of solutions for problems related to fractional and first order Roesser systems with an integral cost functional is studied. The main result in fractional case is a theorem on the existence of optimal control for a linear system considered in the class of absolutely continuous controls of two variables, without any convexity assumption on the integrand describing the cost functional.

1.4.5 Works on controllability theory

In close relation to the theory of optimal control, the problem of controllability of systems remains.

Ordinary and partial systems In the work [1], basing ourself on the Browder theorem on the fixed point of multivalued mapping, we prove two theorems on controllability of an ordinary system: from a moving set to a fixed convex compact set and from a fixed point to the same point.

In [2], using Graves's theorem ([Gra]) and the theorem on global controllability of a linear autonomous system of type (13) ([BBW]), a sufficient condition for the local controllability of a nonlinear autonomous system of this type has been obtained. The investigation of the pointwise controllability of system (12) means the need to consider nonstandard spaces of solutions and controls. In the work [51], results are obtained regarding the global controllability of the linear autonomous system of type (12), based on the mentioned controllability result for the linear autonomous system type (13) and the relationship between linear autonomous systems of type (12) and (13). A theorem on the local controllability of the nonlinear autonomous system of type (12) is also obtained here. The results obtained in [2] and [51] are of the same type like in the case of ordinary systems.

In [54], a bang-bang principle is derived for a linear, nonautonomous system of the form (13) and a theorem on an approximate controllability for such a system considered with piecewise constant controls. Both statements are important from the point of view of practical applications. The proof of the bang-bang principle is based on the notion of the Aumann integral of a multivalued mapping, while the proof of the approximative theorem is based on a generalization (derived in [54]) to the case of functions of two variables of the classical theorem stating that the functions summable on the interval $[a, b]$ and taking their values in a fixed set M, can be approximated in the sense of L^1-metric by piecewise constant functions taking the values in M ([Ba]).

In [22], a bang-bang principle for a system of type (13) which is nonlinear with respect to the control, is derived. In turn, in [24], a bang-bang principle is obtained for a nonautonomous linear system (13), considered with controls with distributed parameters and boundary ones. Here also an approximative theorem for piecewise constant controls is obtained. In the last part of the work, a nonlinear system is considered, whose right side does not depend on partial derivatives of the first order. Under an additional assumption on the linearity of the right side with respect to the control, an approximative bang-bang principle is proved for controls with distributed parameters.

In the paper [58], we propose a certain algorithm of numerical determining of piecewise constant controls with values in a fixed set M, possessing properties implied by approximative theorem, obtained in the work [24]. The construction of the algorithm is based on the proof of the theorem proved in the work [54] and generalizing the corresponding one-dimensional result from the [Ba], mentioned above.

In [25], a bang-bang principle is obtained for a nonautonomous system of the form (12) linear with respect to the state and control. In the case of a system which is linear only with respect to the state, an approximative theorem for piecewise constant controls is obtained. The method used in [25], as in [54], is based on the concept of Aumann integral.

In [60], the problem of „positivity" of the system of the form (12) is examined ([8]). Sufficient conditions are given to ensure that the nonlinear system is positive. Next, it is shown that in the case of a linear system these conditions are also necessary. They mean (in the case of a linear system) that the entries of the matrices on the right side of the system are non-negative.

In paper [62], we investigate the so-called controllability to the set (bounded polyhedron in $\mathbb{R}^n$)([9]) of ordinary nonlinear control system

$$\dot{x} = f(t, x, u), \ t \in [0, T] \text{ a.e.},$$

considered in a class of controls belonging to a bounded polyhedron (functional) spanned on a finite number of fixed controls with vertex in a fixed control. Using the topological degree, we obtained sufficient condition for such a controllability.

Repetitive processes In the work [61], a continuous repetitive process (14) is tested with functions k and l dependent on the control u and with $\varphi = 0$ (function ψ, which occurs in the initial condition is treated as boundary control). Unlike the work [59], this process is studied in the space of absolutely continuous controls (u, ψ)

[8] calling the system (12) positive, we understand that for nonnegative boundary functions and nonnegative control the trajectory and its appropriate partial derivatives are nonnegative

[9] we say that the system is controllable to a set, if it is controllable to each point of this set

and trajectories (z, w). Obtained results on the existence, uniqueness and continuous dependence of solutions on controls allowed to derive the bang-bang principle for the studied process.

In paper [63], we investigate the controllability of linear differential repetitive process of the form

$$\begin{cases} \dot{z}_{k+1}(t) = A_1 z_{k+1}(t) + A_2 w_k(t) + B u_{k+1}(t) \\ w_{k+1}(t) = C_1 z_{k+1}(t) + C_2 w_k(t) + D u_{k+1}(t) \end{cases}$$

for $k \in \mathbb{N} \cup \{0\}$, $t \in \mathbb{R}$, $0 \leq t \leq \alpha$ ($\alpha > 0$ is a fixed real number), with initial conditions

$$\begin{cases} z_k(0) = d_k \text{ for } k \in \mathbb{N}, \\ w_0(t) = f(t) \text{ for } t \in \mathbb{R},\ 0 \leq t \leq \alpha, \end{cases}$$

where $z_k(t) \in \mathbb{R}^n$, $w_k(t) \in \mathbb{R}^m$, $u_k(t) \in \mathbb{R}^r$, coefficients appearing on the right side of the equation are matrices of appropriate dimensions, points $d_k \in \mathbb{R}^n$ and function $f : [0, \alpha] \to \mathbb{R}^n$ are given. In this work, the concept of pointwise controllability of such a process along all passes k is introduced (there is a countable number of all passes) and a theorem on density in space $\prod_{k=1}^{\infty} \mathbb{R}^n$ of the set $\mathcal{A}_{M,PC}$ of points to which one can steer the system using the piecewise constant controls, taking the values in a convex compact set M, in the set $\mathcal{A}_M$ of points to which the system can be steered using measurable controls, taking values in M ($\mathcal{A}_M = \overline{\mathcal{A}_{M,PC}}$).

Paper [27] is a continuation of [63]. Using a result on controllability of abstract control systems, which comes from the work [Kn], we obtained, among others, the following equalities characterizing the pointwise controllability of repetitive processes along all passes k, assuming that the set M of control values is a compact and convex subset of the space $\mathbb{R}^r$: $\mathcal{A}_M = \overline{\mathcal{A}_{exM,PC}}$, where exM stands for the set of extreme points of the set M. It is the counterpart of the classical result of this type for ordinary (finite) control systems.

Also in paper [29], the study of controllability of differential repetitive processes is continued. The result obtained here is an infinite dimensional analogue of the theorem on controllability to the polyhedron, obtained in [62]. While in the [62] the proof of the basic theorem is based on the concept of the topological degree, in this work the infinite dimensional variant of the Poincaré-Miranda theorem is proved and then, basing ourselves on this theorem, we obtain the main result on controllability of repetitive processes to polyhedron (infinitely dimensional). The idea of using the Poincaré-Miranda theorem to study the controllability to the polyhedron comes from the work [BM].

In paper [38], we consider the process

$$\begin{cases} {}^C D^{\alpha}_{a+} z_{k+1}(t) = A_1 z_{k+1}(t) + A_2 w_k(t) + B u_{k+1}(t) \\ w_{k+1}(t) = C_1 z_{k+1}(t) + C_2 w_k(t) + D u_{k+1}(t) \end{cases}$$

with initial conditions

$$\begin{cases} z_k(a) = c_k \text{ for } k \in \mathbb{N}, \\ w_0(t) = f(t) \text{ for } t \in [a, b], \end{cases} .$$

We obtained a theorem saying that the closure of the set of points $(z_k(b))_{k\in\mathbb{N}}$ that can be reached with the aid of controls $\mathfrak{u} = (u_k(\cdot))_{k\in\mathbb{N}} = (I_{a+}^{1-\alpha}v_k(\cdot))_{k\in\mathbb{N}}$ with functions v_k being piecewise constant and taking the values in a fixed convex compact set $M \in \mathbb{R}^r$ is equal to the set of points that can be reached with the aid of controls with summable v_k which take the values in M.

1.4.6 Works on issues of a basic nature

In addition to mainstream research, I also dealt with related issues of a fundamental nature. Beside the aforementioned:

- generalization of the Du Bois-Reymond lemma for functions of one variable to the case of matrix-periodic boundary conditions and derivatives of order k ([14])
- generalization of the Du Bois-Reymond lemma for functions of two variables satisfying Dirichlet boundary conditions to the case of mixed partial derivatives of any order ([12])
- characterization of family of equiabsolutely continuous functions of two variables ([17])
- Helly's principle of choice for functions of two variables of finite variation ([6])
- multidimensional version of theorem on approximation of summable functions by piecewise constant functions ([54])
- infinite dimensional variant of Poincaré-Miranda theorem ([29])

I also studied

- continuity of the superposition (Nemytskii) operator on a subspace ([21])
- differentiability (classical and distributional) of functions of two variables of finite variation ([6], [16])
- diffeomorphisms acting from Banach space to Hilbert space ([35])
- existence and properties of global implicit function ([41], [47])
- generalization of the Du Bois-Reymond lemma for functions of one variable to the case of Riemann-Liouville derivatives of order $\alpha \in (0, 1)$ and higher ([43], [36])
- spaces of Sobolev type of functions possessing Riemann-Liouville derivatives, their properties, imbeddings and compactness of these imbeddings ([34], [65], [39])

- functions of one variable of finite fractional variation and their application to fractional impulsive equations ([48])

In paper [21], a generalization of the classical Krasnoselskii theorem on the superposition operator acting between L^p spaces is proved to the case when this operator is defined on the product of a finite number of subspaces of the space of measurable functions. The obtained result has been used in the work to study the continuous differentiability of the integral functional defined on the product of the Sobolev spaces of the functions of one variable.

In paper [6], the notions of the nondecreasing function of two variables and the function of two variables of finite variation on the interval P have been introduced and theorems on the differentiability (in the sense of Frechet) *a.e.* of such functions have been derived. The problem of definition of a function of two variables of finite variation was taken by many authors. Are known, among others, definitions of Tonelli, Vitalie-Lebesgue-Frechet-de la Vallee Poussin, Frechet, Arzela, Hardy-Krause, Serrin. The starting point for the definition adopted in [6] was the description of an absolutely continuous function of two variables proposed in the work [Wal1]. It can be shown that our definition is equivalent to the one proposed by Hardy and Krause ([CA]) and allows to prove many analogous theorems, as in the case of functions of one variable, among others, the aforementioned theorem on differentiability *a.e.* and Helly's principle of choice ([7]). It is worth adding ([Mos]) that this principle is not true in the class of functions of two variables with finite variation in the sense of Tonelli (the basic problem here is the „size" of the set, where the chosen subsequence is convergent - some characterization of this set, in the case of Tonelli's definition, is given in [Nad]). Similarly, in the case of the property of differentiability - a function of finite variation in the sense of Tonelli may not have differential at any point. Theorem on differentiability *a.e.* of a function of finite variation, derived in [6], can be obtained using a more general result proved in [CC]. In [6], a direct proof of this theorem is presented.

In the work [16], the concept of a function of two variables of locally finite variation has been introduced, and then the theorems characterizing distributional derivatives $D^{(1,1)}, D^{(1,0)}, D^{(0,1)}$ of such a function have been derived. We used the obtained results to prove a theorem on the existence of a solution to the linear partial second order autonomous equation with the right side containing the distributional derivative $D^{(1,1)}$ of a function of two variables of locally finite variation. This theorem is illustrated by an example of the hyperbolic partial impulsive equation, i.e. the equation whose right side contains the Dirac delta concentrated at a fixed point. The inspiration for undertaking research in this direction were the corresponding results for the functions of one variable and ordinary impulsive equations, presented in monograph [HW].

In the work [35], a theorem on the diffeomorphism acting from the Banach space to the Hilbert space, has been derived. The main assumption, in addition to the local invertibility of the operator, is the condition (of variational nature) of Palais-

Smale imposed on a class of appropriate functionals. This theorem has an abstract character and gives a new method for studying equations dependent in a linear way on a parameter ([35], [37], [40]).

In paper [41], the theorem on diffeomorphism has been generalized to a global implicit function theorem, given by equation

$$F(x,y) = 0$$

where $F : X \times Y \to H$ (X, Y are Banach spaces, H - a Hilbert space), y - independent variable (parameter), x - dependent variable. Using this theorem one can examine existence, uniqueness and differentiability of the dependence of solution on parameter for equations involving the parameter nonlinearly ([41], [45]). A certain generalization of this theorem is proved in the paper [47]. The main change in relation to the work [41] consists in replacing the assumption on bijectivity of the operator $F'_x(x,y) : X \to H$ at points (x,y) such that $F(x,y) \neq 0$ by the condition $F(x,y) \notin (\operatorname{Im} F'_x(x,y))^{\perp}$ (ortogonal subspace). In the work, an example is given showing that this condition is fulfilled, while the differential is not bijective.

The main results of work [43] are the Du Bois-Reymond lemma for the functions of one variable possessing Riemann-Liouville derivatives of the order α summable with the power p where $\frac{1}{p} < \alpha < 1$, and a theorem on the integration by parts, taking into account pointwise components, for functions of one variable having right side and left side Riemann-Liouville derivatives of the order α summable with the powers p and q, respectively, assuming that $\frac{1}{p} < \alpha < 1$, $\frac{1}{q} < \alpha < 1$. The special case of this theorem, for functions that are the primary functions of order α of the summable functions (with the appropriate power), is a classical result, which can be found in the monograph [SKM]. Moreover, in the work [43], the characterization (integral representation) of functions having Riemann-Liouville derivatives of the order $\alpha \in (0,1)$ is given. The obtained Du Bois-Reymond lemma of order α can be, as in the case of integer order derivatives, used in the variational method of studying differential equations, to analyze the regularity of a weak solution. The results obtained in [43] have been used to derive the Euler-Lagrange equation together with the boundary conditions for the Bolza functional defined on the space of functions having Riemann-Liouville derivative of order α, and to demonstrate the existence of a solution to the boundary value problem

$$\begin{cases} D^{\alpha}_{b-} D^{\alpha}_{a+} x(t) + x(t) = f(t), \ t \in [a,b] \text{ a.e.} \\ (I^{1-\alpha}_{a+} x)(a) = x_a, \ x(b) = x_b \end{cases},$$

where $\alpha \in (\frac{1}{2}, 1)$, in the space of functions possessing Riemann-Liouville derivative of order α summable with power two.

The results from [43] have been generalized to the case of higher derivatives in paper [36]. More precisely, the Du Bois-Reymond lemma was obtained for derivatives of the order $\alpha \in (n - \frac{1}{2}, n)$, $n \in \mathbb{N}$, $n \geq 2$ and the theorem on the integration by parts - for $\alpha \in (n-1, n)$, $\frac{1}{\alpha - n + 1} < p, q < \infty$. In addition, an integral representation of functions having Riemann-Liouville derivatives of any order α has been derived.

In works [34], [65], imbeddings of spaces $AC_{a+}^{\alpha;p}$ of functions of one variable possessing Riemann-Liouville derivatives of order α summable with power p, have been investigated. In [34], the full characterization of the set of pairs (α, β), for which the relationship

$$AC_{a+}^{\alpha,1} \subset AC_{a+}^{\beta,1}$$

takes place, is given. The main result of the work [65] is the compactness of the above imbeddings.

Work [39] has fundamental character. In this work, the fractional Sobolev spaces of functions of one variable have been introduced using the notion of the Riemann-Liouville derivative. In a special case, namely when $\alpha \in \mathbb{N}$, these spaces reduce to the classical Sobolev spaces of the functions of one variable. In the paper, the weak derivatives of fractional order have been defined with the help of the test functions $\varphi \in C_c^\infty(a, b)$ (functions of class C^∞ with compact supports). It has been shown that these derivatives are identical with Riemann-Liouville derivatives. Two types of norms have been introduced in the fractional spaces and their equivalence was demonstrated. Important results are also imbedding theorems for fractional Sobolev spaces and compactness some of the imbeddings, including compactness of the imbeddings of fractional Sobolev spaces in the spaces of the summable functions. In addition, the completeness, reflexivity and separability of the fractional Sobolev spaces has been proved.

Paper [48], in the part concerning a functional class, can be treated as a continuation of the work [39] (see also [16]). In this work, the concept of a function of finite α-variation is introduced and the characterization of such functions is given. Next, the concept of a weak σ-additive derivative of the order α is introduced and a theorem stating that the summable function has a weak σ-additive derivative of the order α if and only if it has a finite α-variation. In the classical case of $\alpha = 1$, this theorem reduces to the characterization of a function possessing the distributional derivative determined by the σ-additive function of the set. In the second part of the work [48], theorem on the existence of a solution to a linear differential equation with a weak σ-additive derivative of order α of the unknown function, whose right side contains the weak σ-additive derivative of order α of a given function of finite α-variation, is proved. The Cauchy formula for the solution to such an equation, fulfilling the given initial condition, is derived. These results are used to determine the solution of the differential equation with the σ-additive derivative of order α of the unknown function, whose right side contains the Dirac delta concentrated in one or finite number of points. Our paper presents one of two approaches to impulse equations used in the literature, in which the solution is understood as a function of finite variation satisfying the equation in the distributional sense. This approach was also used in [16], following the monograph [HW]. In the case of systems with the fractional derivative, this method was not used by other authors. For the fractional systems an alternative method was used, consisting in determining of the values of jumps of the solution at fixed times, in the form of initial conditions, with the solution understood as a function that is piecewise differentiable ([AABH], [BS], [BHN],

[LBS], [SP]).

1.4.7 Concluding remarks

The most important results include:

- theorem on the diffeomorphism and global implicit function theorem as tools for studying the functional equations ([35], [41])
- theorems on continuous dependence of solutions on right side for a semilinear operator equation ([18])
- definition of fractional Sobolev spaces based on the Riemann-Liouville derivative and demonstration of the identity of fractional distributional derivatives and Riemann-Liouville ones ([39])
- definition of the function of finite α-variation and characterization of its distributional derivative ([48])
- definition of the function of two variables of finite variation and Helly's principle of choice for such functions ([6], [7])
- characterization of distributional derivatives of functions of two variables of locally finite variation ([16])
- generalizations of Du Bois-Reymond lemma ([12], [14], [43])
- generalization of the condition of coercivity on the kernel (4) to the case of a system of order $2k$ ([14])
- infinite dimensional variant of Poincare-Miranda theorem ([29])
- bang-bang principle and approximative theorem in the class of piecewise constant functions for Goursat-Darboux systems ([54])
- maximum principles for ordinary second order systems on bounded and unbounded interval ([15], [30])
- generalization of Krasnoselskii theorem ([21])
- approximative theorem in the class of piecewise constant controls for differential repetitive processes ([27])
- variational asymptotic stability of zero solution to second order system and theorem on such a stability ([28])

1.4.8 References

[AABH] N. Abada, R. P. Agarwal, M. Benchohra, H. Hammouche, *Impulsive semilinear neutral functional differential inclusions with multivalued jumps*, Applications of Mathematics 56 (2) (2011), 227-250.

[ACV] M. Allen, L. Caffarelli, A. Vasseur, *Porous medium flow with both a fractional potential pressure and fractional time derivative*, Chinese Annals of Mathematics 38 (1) (2017), 45-82.

[ALP] S. Ahmad, A. C. Lazer, J. L. Paul, *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance*, Indiana Univ. Math. J. 25 (1976), 933-944.

[AO] A. Alexiewicz, W. Orlicz, *Some remarks on the existence and uniqueness of solutions of the hyperbolic equation* $\frac{\partial^2 z}{\partial x \partial y} = f(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})$, Studia Mathematica XV (2) (1956), 201-215.

[Ba] A. Bacciotti, *Linear systems with piecewise constant controls*, Bolletino U. M. I. (5), 18-A (1981), 102-105.

[BT] R. L. Bagley, P. J. Torvik, *On the fractional calculus model of Viscoelastic behavior,* Journal of Rheology 30 (1) (1986), 133-155.

[BS] D. D. Bainov, P. S. Simeonov, *Systems with Impulsive Effect. Stability, Theory and Applications.* Horwood, Chichister, 1989.

[BBCh] M. Balcerzak, S. A. Belov, V. V. Chistyakov, *On Helly's principle for metric semigroup valued BV mappings of two real variables*, Bulletin of the Australian Mathematical Society 66 (2) (2002), 245-257.

[BHN] M. Benchohra, J. Henderson, S. K. Ntouyas, *Impulsive Differential Equations and Inclusions,* Hindawi Publishing Corporation, New York, 2006.

[BBW] J. Bergmann, P. Burgmeier, P. Weiher, *Explicit controllability criteria for Goursat problems*, Optimization 20 (3) (1989), 281-290.

[BG] E. Berkson, T. A. Gillespie, *Absolutely continuous functions of two variables and well-bounded operators*, J. London Math. Soc. 30 (2) (1984), 305-321.

[BL] S. R. Bernfeld, V. Lakshmikantham, *An Introduction to Nonlinear Boundary Value Problems*, New York, London, 1974.

[BM] Bors, D., M. Majewski, "*On the controllability to the interval of the system governed by a hyperbolic equation*", Kybernetes 38 (7/8) (2009), 1178-1186.

[Bo] L. Bourdin, *Existence of a weak solution for fractional Euler-Lagrange equations*, Journal of Mathematical Analysis and Applications 399 (2013), 239-251.

[CH] D. A. Carlson, A. Haurie, *Infinite Horizon Optimal Control, Theory and Applications*, Lecture Notes in Economics and Mathematical Systems, 290, Springer-Verlag, New York, 1987.

[CC] Y. Chabrillac, J.-P. Crouzeix, *Continuity and differentiability properties of monotone real functions of several variables*, Math. Programming Study 30 (1987), 1–16.

[Ch] V. V. Chistyakov, *Superposition operators in the algebra of functions of two variables*, Monatshefte fur Mathematik 137 (2) (2002), 99-114.

[ChT1] V. V. Chistyakov, Y. V. Tretyachenko, *Selection principles for maps of several variables*, Doklady Mathematics 81 (2) (2010), 282-285.

[ChT2] V. V. Chistyakov, Y. V. Tretyachenko, *Maps of several variables of finite total variation. I. Mixed differences and the total variation*, Journal of Mathematical Analysis and Applications 370 (2) (2010), 672-686.

[ChT3] V. V. Chistyakov, Y. V. Tretyachenko, *Maps of several variables of finite total variation. II. E. Helly-type pointwise selection principles*, Journal of Mathematical Analysis and Applications 369 (1) (2010), 82-93.

[CA] J. A. Clarkson, C. R. Adams, *On definitions of bounded variation for functions of two variables*, Trans. Amer. Math. Soc. 35 (1933), 824-854.

[Ga] R. Gaines, *Continuous dependence for two-point boundary value problems*, Pacific J. Math. 28 (1969), 327-336.

[GR] M. Goebel, U. Raitums, *Constrained control of a nonlinear two point boundary value problem, I*, J. Global Optim. 4 (1994), 367-395.

[Gra] L. M. Graves, *Some mapping theorems*, Duke Math. J. 17 (1950), 111–114.

[HW] A. Halanay, D. Wexler, *Kaczestwiennaja Teoria Impulsnych Sistem*, Moscov, 1971.

[Ing] S. K. Ingram, *Continuous dependence on parameters and boundary data for nonlinear two-point boundary value problems*, Pacific J. Math. 41 (1972), 395-408.

[IT] A. D. Ioffe, V. M. Tikhomirov, *Theory of Extremal Problems*, North-Holland, 1979.

[Kam] R. Kamocki, *On the existence of optimal solutions to fractional optimal control problems*, Appl. Math. Comput. 235 (2014), 94-104.

[Kis] M. Kisielewicz, Differential Inclusions and Optimal Control, Kluwer Academic Publishers, Dordrecht, PWN, Warszawa, 1991.

[Kla] G. Klassen, *Dependence of solutions on boundary conditions for second order ordinary differential equations*, J. Differential Equations 7 (1970), 24-33.

[Kn] G. Knowles, *Lyapunov vector measures*, SIAM J. Control 13 (2) (1975), 294-303.

[LBS] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, *Theory of Impulsive Differential Equations*, Worlds Scientific, Singapore, 1989.

[La] T. A. M. Langlands, *Solution of a modified fractional diffusion equation*, Physical A, Statistical Mechanics and its Applications 367 (2006), 136-144.

[Le] H. Leiva, *Existence of bounded solutions of a second-order system with dissipation*, J. Math. Anal. Appl. 237 (1999), 288-302.

[LP] A. Lepin, W. Ponomariev, *Continuous dependence of solutions of the boundary value problem for ordinary differential equations*, Diferencialnyje Urawnienija 9 (4) (1973), 626-629.

[MS] E. A. Markin, A. S. Strekalowskii, *O suszcziestwowanii, jedinstwiennosti i ustojcziwosti rieszenia dla odnowo klasa dinamiczeskich sistem opisywajuszczich chimiczeskije procesy*, Vestnik Moskovskogo Universiteta, Seria Vyczislitelnoj Matematiki i Kibernetiki 4 (1977), 3–11.

[Maw] J. Mawhin, *Metody Wariacyjne dla Nieliniowych Problemów Dirichleta*, WNT, Warszawa, 1994.

[MW] J. Mawhin, M. Willem, *Critical Point Theory and Hamiltonian Systems,* Springer-Verlag, New York, 1989.

[MZ] J. Mawhin, F. Zanolin, *A continuation approach to fourth order superlinear periodic boundary value problems*, Topological Methods in Nonlinear Analysis 2 (1) (1993), 55-74.

[Mos] P. P. Mosolov, *O problemie minimuma funkcjonała*, Izwestija AN SSSR Ser. Matem. 31 (6) (1967), 1289–1310.

[Nad] N. S. Nadiraszwili, *Princip wybora Helly dla funkcij dwóch peremennych*, Westnik Moskowsk. Uniwersiteta, Matematika 3 (1975), 3–10.

[OZ] P. Omari, F. Zanolin, *On forced nonlinear oscillations in n-th order differential systems with geometric conditions*, Nonlinear Anal. 8 (1984), 723-748.

[Pie] W. Pieczka, *On the general solution of some partial differential equations in two–variables domain*, Demonstratio Mathematica VII (4) (1974), 451-462.

[PS1] W. I. Płotnikov, W. I. Sumin, *Problemy ustojcziwosti nieliniejnych system Goursat-Darboux*, Diferencjalnyje Urawnienija VIII (5) (1972), 845–856.

[PS2] W. I. Płotnikov, W. I. Sumin, *Optimizacja obiektow z raspriedielonnymi parametrami, opisywajemych sistemami Goursat-Darboux*, Żurnal Wyczislitielnoj Matematiki i Matematiczeskoj Fiziki 12 (1) (1972), 61–77.

[PAT] S. Pooseh, R. Almeida, D. F. M. Torres, *Fractional order optimal control problems with free terminal time*, J. Ind. Manage. Optim. 10 (2014), 363-381.

[SKM] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives - Theory and Applications*, Gordon and Breach, Amsterdam, 1993.

[SP] A. M. Samoilenko, N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[Se1] S. Sędziwy, *Dependence of solutions on boundary data for a system of two ordinary differential equations*, J. Differential Equations 9 (1971), 381-389.

[Se2] S. Sędziwy, *Bounded solutions of second order systems of differential equations*, J. Math. Anal. Appl. 281 (2003), 697-706.

[Sr] W. A. Sroczko, *Uslowija optimalnosti tipa principa maksimuma w sistemach Goursat-Darboux*, Sibirskij Matematiczeskij Żurnal XXV (1) (1984), 126–132.

[Sur1] M. B. Suryanarayana, *Necessary conditions for optimization problems with hyperbolic partial differential equations*, SIAM J. Control 11 (1) (1973), 130–147.

[Sur2] M. B. Suryanarayana, *Existence theorems for optimization problems concerning hyperbolic partial differential equations*, Journal of Optimization Theory and Applications 15 (4) (1975), 361-392.

[Sur3] M. B. Suryanarayana, *A Sobolev space and a Darboux problem*, Pacific J. Math. 69 (2) (1977), 535-550.

[TS] A. N. Tichonov, A. A. Samarskii, *Urawnienija Matematiczeskoj Fiziki*, Moskwa, 1964.

[VK] N. Vasiliev, J. Klokov, *Osnowy Teorii Granicznych Zadacz dla Obyknowiennych Difieriencialnych Urawnienij,* Riga, 1978.

[VS] O. V. Vasiliew, V. A. Sroczko, K optimizacii odnowo klasa upravlaemych procesov s raspredelonnymi parametrami, Sib. Mat. J. XIX (2) (1978).

[VG] A. N. Vityuk and A. V. Golushkov, *Existence of solutions of systems of partial differential equations of fractional order*, Nonlinear Oscillations 7 (3) (2004), 318–325.

[Wal1] S. Walczak, *Absolutely continuous functions of several variables and their applications to differential equations*, Bull. Polish Acad. Sci. 35 (11-12) (1987), 733-744.

[Wal2] S. Walczak, *On the Du Bois-Reymond lemma for functions of several variables*, Bull. Soc. Math. Belg. Ser. B 45 (3) (1993), 225-235.

[Wal3] S. Walczak, *On the continuous dependence on parameters of solutions of the Dirichlet problem. Part I. Coercive case, Part II. The case of saddle points*, Bulletin de la Classe des Sciences de l'Academie Royale de Belgique 6 (7–12) (1995), 247–273.

[WE] S. Westerlund, L. Ekstam, *Capacitor theory*, IEEE Trans. actions on Dielectics and Electrical Insulation 1 (5) (1994), 826-839.

[YGD] H. Ye, J. Gao, Y. Ding, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl. 328 (2007), 1075-1081.

1.5 Citation of papers

48 publications are included in the MathSciNet database. These works are cited 148 times (115 times without self-citations) by 82 authors. The Hirsch index is $H = 8$.

43 publications are included in the Scopus database. These works are cited 247 times (187 times without self-citations) in 163 documents. The Hirsch index is $H = 10$.

Below, a list of the cited works is given, taking into account the number of citations according to the above-mentioned databases (in parentheses, the number of citations without self-citations is given). The numbering of works is in accordance with the numbering adopted in chapters 1.1 and 1.2.

[35] D. Idczak, A. Skowron, S. Walczak, *On the diffeomorphisms between Banach and Hilbert spaces*, Advanced Nonlinear Studies 12 (2012), 89–100.

MathSciNet: **17** (**13**)

Scopus: **24** (**20**)

[33] D. Idczak, R. Kamocki, *On the existence and uniqueness and formula for the solution of R-L fractional Cauchy problem in* $\mathbb{R}^n$, Fractional Calculus and Applied Analysis 14 (4) (2011), 538-553.

MathSciNet: **11** (**7**)

Scopus: **20** (14)

[18] D. Idczak, *Stability in semilinear problems*, J. Differential Equations 162 (2000), 64-90.

MathSciNet: **19** (**19**)

Scopus: **18** (**18**)

[21] D. Idczak, A. Rogowski, *On a generalization of Krasnoselskii's theorem*, J. Austral. Math. Soc. 72 (2002), 389-394.

MathSciNet: **10** (**9**)

Scopus: **18** (**17**)

[7] D. Idczak, S. Walczak, *On Helly's theorem for functions of several variables and its applications to variational problems*, Optimization 30 (1994), 331-343.

MathSciNet: **10** (**10**)

Scopus: **16** (**15**)

[36] D. Idczak, M. Majewski, *Fractional fundamental lemma of order* $\alpha \in (n - 1/2, n)$ *with* $n \in \mathbb{N}$, $n \geq 2$, Dynamic Systems and Applications 21 (2012), 251-268.

MathSciNet: **8** (**4**)

Scopus: **15** (**8**)

[39] D. Idczak, S. Walczak, *Fractional Sobolev spaces via Riemann-Liouville derivatives,* Journal of Function Spaces and Applications, vol. 2013, Article ID 128043, 15 pages, http://dx.doi.org/10.1155/2013/128043.

MathSciNet: **7** (**6**)

Scopus: **13** (**12**)

[43] L. Bourdin, D. Idczak, *A fractional fundamental lemma and fractional integration by parts formula - Applications to critical points of Bolza functionals and to linear boundary value problems*, Advances in Differential Equations 20 (3-4) (2015), 213-232.

MathSciNet: **9** (**3**)

Scopus: **12** (**8**)

[15] D. Idczak, *Optimal control of a coercive Dirichlet problem*, SIAM J. Control Optim. 36 (4) (1998), 1250-1267.

MathSciNet: **9** (**6**)

Scopus: **10** (**6**)

[41] D. Idczak, *A global implicit function theorem and its applications to functional equations*, Discrete and Continuous Dynamical Systems, Series B 19 (8) (2014), 2549-2556.

MathSciNet: **7** (**5**)

Scopus: **10** (**8**)

[37] D. Idczak, A. Skowron, S. Walczak, *Sensitivity of a fractional integro-differential Cauchy problem of Volterra type*, Abstract and Applied Analysis, vol. 2013, Article ID 129478, 8 pages, http://dx.doi.org/10.1155/2013/129478.

MathSciNet: **7** (**6**)

Scopus: **7** (**6**)

[22] D. Idczak, *Bang-bang principle for linear and nonlinear Goursat-Darboux problems*, Int. J. Control 76 (11) (2003), 1089-1094.
MathSciNet: **3** (**3**)
Scopus: **7** (**5**)

[38] D. Idczak, R. Kamocki, *Fractional differential repetitive processes with Riemann-Liouville and Caputo derivatives*, Multidimensional Systems and Signal Processing 26 (2015), 193-206, DOI 10.1007/s11045-013-0249-0, published online: 25.09.2013.

MathSciNet: **3** (**1**)

Scopus: **7** (**5**)

[34] D. Idczak, S. Walczak, *A fractional imbedding theorem*, Fractional Calculus and Applied Analysis 15 (3) (2012), 418-425.

MathSciNet: **3** (**1**)

Scopus: **6** (**2**)

[2] D. Idczak, S. Walczak, *On the controllability of nonlinear Goursat systems*, Optimization 23 (1992), 91-98.

MathSciNet: **0** (**0**)

Scopus: **6** (**5**)

[24] D. Idczak, S. Walczak, *On some properties of Goursat-Darboux systems with distributed and boundary controls*, Int. J. Control 77 (9) (2004), 837-846.

MathSciNet: **1** (**1**)

Scopus: **5** (**3**)

[46] D. Idczak, S. Walczak, *On a linear-quadratic problem with Caputo derivative*, Opuscula Mathematica 36 (1) (2016), 49-68.

MathSciNet: **3** (**3**)

Scopus: **4** (**4**)

[47] D. Idczak, *On a generalization of a global implicit function theorem*, Advanced Nonlinear Studies 16 (1) (2016); DOI: 10.1515/ans-2015-5008.

MathSciNet: **1** (**0**)

Scopus: **4** (**3**)

[6] D. Idczak, *Functions of several variables of finite variation and their differentiability*, Annales Polonici Mathematici LX.1 (1994), 47-56.

MathSciNet: **4** (**4**)

Scopus: the paper is not included in the database

[13] D. Idczak, *Necessary optimality conditions for a nonlinear continuous n-D Roesser model*, Mathematics and Computers in Simulation 41 (1996), 87-94.

MathSciNet: **0** (**0**)

Scopus: **4** (**2**)

[29] D. Idczak, M. Majewski, *A generalization of the Poincare-Miranda theorem with an application to the controllability of nonlinear repetitive processes*, K. Gałkowski and E. Rogers (Eds.): Recent Developments on Multidimensional Systems, Control and Signals – Theory and Applications, ASIAN J. Control 12 (2) (2010), 1-9.

MathSciNet: **2** (**2**)

Scopus: **3** (**3**)

[23] D. Idczak, M. Majewski, S. Walczak, *Stability analysis of solutions to an optimal control problem associated with a Goursat-Darboux problem*, K. Gałkowski, R. W. Longman, E. Rogers (Eds.): Multidimensional Systems nD and Iterative Learning Control, Int. J. Appl. Math. Comput. Sci. 13 (1) (2003), 29-44.

MathSciNet: **3** (**3**)

Scopus: the paper is not included in the database

[45] D. Idczak, S. Walczak, *Application of a global implicit function theorem to a general fractional integro-differential system of Volterra type*, Journal of Integral Equations and Applications 27 (4) (2015), 521-554.

MathSciNet: **1** (**1**)

Scopus: **2** (**2**)

[49] T. Kaczorek, D. Idczak, *Cauchy formula for the time-varying linear systems with Caputo derivative*, Fractional Calculus and Applied Analysis 20 (2) (2017), 494-505; DOI: 10.1515/fca-2017-0025.

MathSciNet: **1** (**1**)

Scopus: **2** (**2**)

[14] D. Idczak, *M-periodic problem of order 2k*, Topological Methods in Nonlinear Analysis 11 (1998), 169-185.

MathSciNet: **2** (**2**)

Scopus: the paper is not included in the database

[17] D. Idczak, S. Walczak, *On the existence of a solution for some distributed optimal control hyperbolic system*, Internat. J. Math. & Math. Sci., vol. 23, no. 5 (2000), 297-311.

MathSciNet: **2** (**2**)

Scopus: the paper is not included in the database

[25] D. Idczak, M. Majewski, *Bang-bang controls and piecewise constant ones for continuous Roesser system*, Multidimensional Systems and Signal Processing 17 (2006), 243-255.

MathSciNet: **0** (**0**)

Scopus: **2** (**2**)

[32] D. Idczak, M. Majewski, *Existence of optimal solutions of two-directionally continuous linear repetitive processes*, Multidimensional Systems and Signal Processing 23 (2012), 155-162, DOI: 10.1007/s11045-010-0105-4; published online: 7.04.2010.

MathSciNet: **0** (**0**)

Scopus: **2** (**2**)

[26] D. Idczak, *Maximum principle for optimal control of two-directionally continuous linear repetitive processes*, E. Zerz and K. Galkowski (Eds.): Recent Advances in Multidimensional Systems and Signals, Multidimensional Systems and Signal Processing 19 (2008), 411-423.

MathSciNet: **0** (**0**)

Scopus: **2** (**1**)

[16] D. Idczak, *Distributional derivatives of functions of two variables of finite variation and their application to an impulsive hyperbolic equation*, Czechoslovak Mathematical Journal 48 (123) (1998), 145-171.

MathSciNet: **1** (**0**)

Scopus: **1** (**0**)

[42] D. Idczak, S. Walczak, *Optimization of a fractional Mayer problem – existence of solutions, maximum principle, gradient methods*, Opuscula Mathematica 34 (4) (2014), 763-775.

MathSciNet: **1** (**0**)

Scopus: **1** (**0**)

[1] D. Idczak, *Applications of the fixed point theorem to problems of controllability*, Bulletin de la Societe des Sciences et des Lettres de Lodz, vol. XXXIX.3, no. 57 (1989), 1-7.

MathSciNet: **1** (**1**)

Scopus: the paper is not included in the database

[8] D. Idczak, K. Kibalczyc, S. Walczak, *On an optimization problem with cost of rapid variation of control*, J. Austral. Math. Soc. Ser. B 36 (1994), 117-131.

MathSciNet: **1** (**1**)

Scopus: the paper is not included in the database

[9] D. Idczak, S. Walczak, *On the existence of the Carathodory solutions for some boundary value problems*, Rocky Mountain J. Math. 24 (1) (1994), 1-13.

MathSciNet: **0** (**0**)

Scopus: **1** (**1**)

[55] D. Idczak, M. Majewski, *Nonlinear positive 2D systems and optimal control*, L. Benvenuti, A. De Santis and L. Farina (Eds.): Positive Systems - Proceedings of the First Multidisciplinary International Symposium on Positive Systems: Theory and Applications (POSTA 2003), Lecture Notes in Control and Information Sciences 294 (2003), 329-336.

MathSciNet: **1** (**1**)

Scopus: the paper is not included in the database

[27] D. Idczak, *Approximative piecewise constant bang-bang principle for differential repetitive processes*, Int. J. Control 82 (5) (2009), 910-917.

MathSciNet: **0** (**0**)

Scopus: **1** (**0**)

1.6 Research projects

1.6.1 Managing research projects

1. The manager in the research project NCN Nr DEC-2011/01/B/ST7/03426 „*One-dimensional and two-dimensional optimal control systems of noninteger order*", 2011-2014

1.6.2 Participation in research projects

1. The contractor in the research project KBN 2 1102 91 01 „*Systems described by ordinary and partial equations and their optimization*", 1991-1994

2. The contractor in the research project KBN 8T 11A01109 „*Two-dimensional continuous systems and Dirichlet systems and their optimization*", 1995-1998

3. The contractor in the research project KBN 2 PO 3A05910 „*Variational methods in studying of differential equations: existence, dependence on parameters and simulation of solutions*", 1996-1998

4. The contractor in the research project KBN 7 T11A 004 21 „*Investigation of continuous n-dimensional systems , existence of solutions and stability of optimal processes*", 2001-2004

5. The contractor in the research project MNiSW N514 027 32/3630 „*Investigation of continuous and discerete-continuous systems with control"*, 2007-2010

1.7 Participation in scientific conferences (with presentation)

1.7.1 List of conferences

1. The Twelfth International Conference on Nonlinear Oscillations, Cracov, Poland, 1990

2. Dwudziesta Ogólnopolska Konferencja Zastosowań Matematyki, Zakopane-Kościelisko, Polska, 1991

3. Dwudziesta Druga Ogólnopolska Konferencja Zastosowań Matematyki, Zakopane-Kościelisko, Polska, 1993

4. International Conference - Optimal Control of Differential Equations, Athens, USA, 1993

5. 16th IFIP Conference on System Modelling and Optimization, Compiegne, France, 1993

6. International Symposium on Signal Processing, Robotics and Neural Networks, Lille, France, 1994

7. The International Conference on the Theory and Methods of Optimization and their Applications, Spała, Poland, 1994
8. Minisemestr - Topological and Variational Methods of Nonlinear Analysis, International Stefan Banach Center, Warsaw, Poland, 1994
9. Second International Symposium on Methods and Models in Automation and Robotics, Międzyzdroje, Poland, 1995
10. The Second World Congress of Nonlinear Analysts, Athens, Greece, 1996
11. Third International Symposium on Methods and Models in Automation and Robotics, Międzyzdroje, Poland, 1996
12. Workshop on Manufacturing Systems: Modelling, Management and Control, Vien, Austria, 1997
13. Sixth International Colloquium on Numerical Analysis and Computer Sciences with Applications, Plovdiv, Bulgaria, 1997
14. I Forum Równań Różniczkowych, Będlewo, Polska, 1998
15. The First International Workshop on Multidimensional (nD) Systems, Łagów, Poland, 1998
16. International Congress of Mathematicians, Berlin, Germany, 1998
17. 2nd Symposium on Nonlinear Analysis, Toruń, Poland, 1999
18. The Third World Congress of Nonlinear Analysts, Catania, Italy, 2000
19. The Second International Workshop on Multidimensional (nD) Systems, Czocha, Poland, 2000
20. 8th IEEE International Conference on Methods and Models in Automation and Robotics, Szczecin, Poland, 2002
21. 15th International Symposium on the Mathematical Theory of Networks and Systems, University of Notre Dame, Indiana, USA, 2002
22. First Multidisciplinary International Symposium on Positive Systems: Theory and Applications (POSTA 2003), Rome, Italy, 2003
23. Sixteenth International Symposium on Mathematical Theory of Networks and Systems, Leuven, Belgium, 2004
24. IV-te Polskie Sympozjum Analizy Nieliniowej, Łódź, Polska, 2004

25. Fourth International Workshop on Multidimensional (nD) Systems NDS 2005, Wuppertal, Germany, 2005

26. 17th International Symposium on Mathematical Theory of Networks and Systems, Kyoto, Japan, 2006

27. 2007 International Workshop on Multidimensional (nD) Systems, Aveiro, Portugal, 2007

28. 23rd IFIP TC 7 Conference on System Modelling and Optimization, Cracov, Poland, 2007

29. Fifth World Congress of Nonlinear Analysts, Orlando, USA, 2008

30. 19th International Symposium on Mathematical Theory of Networks and Systems, Budapest, Hungary, 2010

31. II Sesja Naukowa - Ułamkowy Rachunek Różniczkowy i Jego Zastosowania, Częstochowa, Polska, 2010

32. III Seminarium Naukowe Rachunek Różniczkowy Niecałkowitego rzędu i Jego Zastosowania, Białystok, Polska, 2011

33. IV Seminarium Naukowe Rachunek Różniczkowy Niecałkowitego Rzędu i Jego Zastosowania, Warszawa, Polska, 2012

34. 17th IEEE International Conference on Methods and Models in Automation and Robotics MMAR 2012, Międzyzdroje, Poland

35. 18th IEEE International Conference on Methods and Models in Automation and Robotics MMAR 2013, Międzyzdroje, Poland

36. Dynamical Systems and Applications, The Conference in Honor of Prof. Avner Friedman, Lodz, Poland, 2013

37. V Konferencja Naukowa Rachunek Różniczkowy Niecałkowitego Rzędu i Jego Zastosowania, Cracow, Poland, 2013

38. Dynamical Systems with Applications II, Lodz, Poland, 2014

39. VI Konferencja Naukowa Rachunek Różniczkowy Niecałkowitego Rzędu i Jego Zastosowania, Opole, Polska, 2014

40. 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications, Madrid, Spain, 2014

41. Joint Meeting of the German Mathematical Society and the Polish Mathematical Society, Poznań, Poland, 2014

42. 3rd Conference on Dynamical Systems and Applications, Lodz, Poland, 2015
43. 7th Conference on Non-Integer Order Calculus and Its Applications, Szczecin, Poland, 2015
44. VII Symposium on Nonlinear Analysis, Toruń, Poland, 2015
45. Equadiff 2017, Bratyslava, Slovakia, 2017
46. The 12th AIMS Conference on Dynamical Systems, Differential Equations and Applications, Taipei, Taiwan, 2018

1.7.2 Plenary and invited lectures, organizing of conference sessions

- Plenary lecture „*Absolutely continuous functions of two variables and functions of finite variation and their application to partial differential equations*", IV-te Polskie Sympozjum Analizy Nieliniowej, Łódź, Polska, 2004
- Invited lecture „*Some continuous 2D systems and their applications*", Sixth International Colloquium on Numerical Analysis and Computer Science with Applications, Plovdiv, Bulgaria, 1997
- Invited lecture „*Stability analysis of two-dimensional optimal control systems*", The Third World Congress of Nonlinear Analysts, Catania, Italy, 2000
- Invited lecture „*Controllability of repetitive processes*", Fifth World Congress of Nonlinear Analysts, Orlando, USA, 2008
- Invited lecture „*A global implicit function theorem and its applications*", 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications, Madrid, Spain, 2014
- Invited lecture „*A bipolynomial fractional Dirchlet-Laplace problem*", The 12th AIMS Conference on Dynamical Systems, Differential Equations and Applications,Taipei, Taiwan, 2018
- Organizing (jointly with Marek Majewski) of the invited session „*2D systems*" during 19th International Symposium on Mathematical Theory of Networks and Systems, Budapest, Hungary, 2010

1.7.3 Lectures at foreign research centers

- Southern Illinois University at Edwardsville, Illinois, USA, 1993
- Washington University, St. Louis, Missouri, USA, 1993

2 Achievements in the field of scientific care and education of young staff

2.1 Supervisor in completed PhDs

1. Marek Majewski, *Stability of solutions to differential systems and control systems*, Uniwersytet Łódzki, 2003
2. Dominika Bogusz, *Optimization of Goursat-Darboux systems with infinite horizon*, Uniwersytet Łódzki, 2008
3. Rafał Kamocki, *Some fractional ordinary and distributed parameters control systems and their optimization*, Uniwersytet Łódzki, 2012 - dissertation awarded by Council of the Faculty of Mathematics and Computer Science UL

2.2 Supervisor in currently open PhDs

1. Kamil Pajek, *Optimal control of differential, integral and integro-differential systems of fractional order,* Uniwersytet Łódzki

2.3 Reviews of PhD thesis

1. Marek Galewski, *Two-point problems for some classes of nonlinear operator equations*, Uniwersytet Łódzki, 2002
2. El Desouky Ramo, *Strongly subregular functions and their applications in optimization*, Uniwersytet Łódzki, 2004
3. Andrzej Skowron, *Critical points of semi-coercive type and their applications*, Uniwersytet Łódzki, 2005
4. Marjan Jakšto, *Elliptic systems with control on unbounded sets*, Uniwersytet Łódzki, 2006
5. Katarzyna Szymańska, *Some asymptotic boundary problems for ordinary differential equations*, Uniwersytet Łódzki, 2006
6. Jan Pustelnik, *Approximation of optimal value of Bolza functional*, Uniwersytet Łódzki, 2009
7. Anna Michalak, *Dual approach to Lyapunov stability problem*, Uniwersytet Łódzki, 2009
8. Mariusz Jurkiewicz, *Resonance boundary Lidstone problem for differential equations of higher orders*, Uniwersytet Łódzki, 2009

9. Anna Kulig, *Nonlinear evolution inclusions and hemivariational inequalities for nonsmooth problems in contact mechanics*, Uniwersytet Jagielloński, 2009

10. Adrian Karpowicz, *Caratheodory solutions to hyperbolic equations and differential functional inequalities*, Uniwersytet Gdański, 2010

11. Jiangfeng Han, *Evolutionary multivalued hemivariational inequalities modeling dynamic unilateral contact problems*, Jagiellonian University, 2016

12. Igor Kossowski, *Boundary problems for ordinary differential equations with nonlocal nonlinear condition*s, Politechnika Łódzka, 2018

2.4 Reviews in habilitation procedures

1. Dorota Bors, Uniwersytet Łódzki, 2015

2. Anna Sikorska-Nowak, Uniwersytet Zielonogórski, 2016

3. Ewa Girejko, Politechnika Białostocka, 2018

3 Activities promoting science

3.1 Translation of scientific works

1. Translation to Polish language of the monograph: J. Mawhin, *Problemes de Dirichlet Variationnels Non Linéaires*, Les Presses de l'Université de Montréal, 1987 (jointly with A. Nowakowski and S. Walczak); bibliographic data of the polish translation: Jean Mawhin, *Metody Wariacyjne dla Nieliniowych Problemów Dirichleta*, WNT, Warszawa, 1994.

3.2 Other forms of activities promoting science

1. Lecture for students and teachers „*On simple optimization problems*" organized by Zarząd Oddziału Łódzkiego PTM, 2002

2. Review lecture „*Some issues from optimal control theory for hyperbolic and elliptic systems*" delivered during Uroczysta Sesja Naukowa z okazji 10-lecia Wydziału Matematyki UŁ, 2006

3. Lecture „*Fractional differential calculus*" within the cycle „Okno na podwórze" organized by Zarząd Oddziału Łódzkiego PTM, 2010

4. Lecture „*Riemann-Liouville derivatives and fractional functional spaces"* within Seminarium Instytutu Matematyki Politechniki Łódzkiej, 2013

5. Lecture „*On some generalized fractional Mayer problem with Caputo derivative*" within Seminarium Zastosowań Matematyki Oddziału PTM w Krakowie, 2014