<u>SCIENTIFIC REPORT</u>

of a candidate for the title of professor

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Academic CV

I was born on March 17, 1961 in Gana, a village in Praszka commune in Lodz voivodeship.

I graduated in Mathematics in the Faculty of Mathematics, Physics and Chemistry at the University of Lodz in 1986.

On 28 June 1994 I got PhD in mathematics in the Faculty of Mathematics, Physics and Chemistry at the University of Lodz, based on the dissertation under the title

Multivalued algebraic mappings and factorization of polynomials,

prepared under the supervision of prof. dr. hab. Jack Chądzyński from the University of Lodz.

Reviewers of this dissertation were:

Prof. dr. Arkadiusz Płoski from the Kielce University of Technology,

prof. dr. Tadeusz Winiarski from the Jagiellonian University.

On the basis of the dissertation:

Local and global properties of analytic mappings and sets in the context of the Łojasiewicz exponent.

I obtained habilitation degree in the Faculty of Mathematics, University of Lodz on 29 March 2006.

The reviewers were:

prof. dr hab. Jacek Chądzyński from the University of Lodz,

prof. dr hab. Tadeusz Mostowski from the Warsaw University,

prof. dr hab. Piotr Tworzewski from the Jagiellonian University.

The degree has been approved by the Central Commission for Academic Degrees and Titles.

Since 15 December 1985 I have worked continuously at the University of Lodz and since 2006 I have been an associate professor of the University in the Department of Analytic Functions and Differential Equations in the Faculty of Mathematics and Computer Science.

Scientific achievements

I will start the description of my most important scientific achievements from the location of the research in comparison to other results. Then I will discuss the results before and after my habilitation. My own (and joint with other authors) important results will be formulated in the form of theorems (12 theorems before the habilitation and 20 – after the habilitation).

My own papers and joint authorship are denoted by numbers in square brackets, and the works cited – by a few first letters of the names of authors in square brackets.

I situate my scientific research in real and complex analytic and algebraic geometry. Before my habilitation these studies were associated rather with complex geometry, and after the habilitation - with real geometry. Most of them focused on the metric properties of algebraic and semialgebraic sets and polynomial nad semialgebraic mappings. Themes in most of my research papers are Łojasiewicz inequalities and exponents for mapping and sets in neighborhoods of a point, neighbourhoods of infinity and global inequalities.

Łojasiewicz inequalities emerged in the late 1950s as the main tool in the division of distributions by functions, which was posed by an outstanding French mathematician L. Schwartz (1957). It was solved by a Swedish mathematician and winner of the Fields Medal L. Hörmander [Hö] (in the case of division of distributions by real polynomials) and by an outstanding Polish mathematician S. Łojasiewicz [$Ł_3$, $Ł_4$] (in the case of division by real analytic functions). To solve this problem Łojasiewicz introduced a subtle and difficult theory of semianalytic sets. This theory has contributed to the development of subanalytic geometry being studied by such prominent mathematicians as A. M. Gabrielov [Ga_1], E. Bierstone and P.D. Milman [BM] and winners of the Fields Medal H. Hironaka [Hir_1, Hir_2] and R. Thom. The Łojasiewicz inequalities play a special role in this theory. They have turned out to be of nontrivial use in numerous branches of mathematics, including differential equations (L. Simon [Sim]), dynamical systems (J. Bolte, A. Daniilidis, O. Ley and L. Mazet [BDLM]) and singularity theory (K. Kurdyka, T. Mostowski and A. Parusiński [KMP]). Quantitative versions of these inequalities, involving e.g. computing or estimating the relevant exponents, are of importance in real and complex algebraic geometry (see results of a known French mathematician B. Teissier [Te]). Recently a strong demand for explicit estimates of the Łojasiewicz exponent comes from optimization theory (see for instance articles of M. Schweighofer [$Schw_3$] and the applicant, joint with J. Kurtyka [37], which is more precisely described in point II.4.c)) and also from estimates for global error bounds (G. Li, B.S. Mordukhovich and T.S. Pham [LMP]).

At first let us quote definitions and basic facts related to these exponents. S. Łojasiewicz used the following inequality while solving the problem of dividing distribution. It is also key point in many other issues theory of semianalytic sets. [$Ł_2$, $Ł_4$]:

For semianalytic and closed sets $X, Y \subset \mathbb{R}^n$ and a point $x_0 \in X \cap Y$ there exist a neighbourhood $\Omega \subset \mathbb{R}^n$ of the point x_0 and positive constants $C, \nu > 0$ such that

$$\text{(S)} \qquad \operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geqslant C \operatorname{dist}(x, X \cap Y)^\nu \quad \text{for} \quad x \in \Omega,$$

where $\operatorname{dist}(x, V)$ is the distance of a point x from a set V ($\operatorname{dist}(x, V) = 1$ for $V = \emptyset$). The condition (S), known as regular separation condition is fulfilled also for subanalytic sets ([Hir_2, $Ł_4$, $Ł_6$]). The smallest exponent ν in (S) is called the *Łojasiewicz exponent in the* *regular separation of sets* X, Y at the point x_0 and denoted by $\mathcal{L}_{x_0}(X,Y)$. It is known that $\mathcal{L}_{x_0}(X,Y) \in \mathbb{Q}$ and that the inequality (S) also holds for $\nu = \mathcal{L}_{x_0}(X,Y)$, some constant $C > 0$ and a neighbourhood Ω of x_0. Similar properties go for the next exponents considered below.

Let $f : (\mathbb{R}^n, a) \to (\mathbb{R}, 0)$ be a real analytic function[1] (or semialgebraic of class C^1) in $x = (x_1, \dots, x_n)$, and let ∇f be the gradient of f, i.e., $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) : (\mathbb{R}^n, a) \to \mathbb{R}^n$. Then there exist positive constants C, ε and a constant $\varrho \in [0,1)$ such that the following *Łojasiewicz gradient inequality* holds (cf. [Ł$_4$] lub [Ł$_5$]):

$$(\text{Ł}_1) \qquad |\nabla f(x)| \geqslant C|f(x)|^{\varrho} \quad \text{for} \quad |x-a| < \varepsilon.$$

The lower bound of the set of exponents ϱ in (Ł$_1$), denoted by $\varrho_a(f)$, is called the *Łojasiewicz exponent in the gradient inequality.* Łojasiewicz used (Ł$_1$) to show that the set of zeros of an analytic function f in a neighborhood of the point a is a deformational retract of the neighbourhood [Ł$_5$]. It is closely associated with the gradient hypothesis of R. Thom (see [KMP]) and also holds for holomorphic functions $f : (\mathbb{C}^n, a) \to (\mathbb{C}, 0)$, see [L-JT]. Knowledge of an estimate of the exponent $\varrho_a(f)$, allows us to estimate the Łojasiewicz exponents of function f and the gradient of the function f (see [35]).

If $F : (\mathbb{R}^n, a) \to (\mathbb{R}^m, 0)$ is a real analytic mapping or a continuous semialgebraic one (defined in a neighbourhood $U \subset \mathbb{R}^n$ of a), then there exist positive constants C, η, ε for which the following *Łojasiewicz ibequality* holds:

$$(\text{Ł}_2) \qquad |F(x)| \geqslant C \operatorname{dist}(x, V(F))^{\eta} \quad \text{for} \quad |x-a| < \varepsilon,$$

where $V(F) = \{x \in U : F(x) = 0\}$. The lower bound of the set of exponents η in (Ł$_2$), denoted by $\mathcal{L}_a(F)$, is called the *Łojasiewicz exponent* of the mapping F at the point a. The exponent $\mathcal{L}_a(F)$ is an important invariant and tool in the singularity theory. It is associated among other things with the Noether exponent, multiplicity of a mapping at a point, the Milnor number and differential equations theory (see for instance [AK$_1$, AK$_2$, BŁ, BR, Cy$_3$, Ga$_2$, Gw, Ko$_1$, Ko$_2$, Ko$_3$, Kui, Kuo$_1$, Kuo$_2$, KMP, KP, L-JT, Pł$_2$, 19, Te]).

By the *Łojasiewicz exponent at infinity* of a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, denoted by $\mathcal{L}_\infty(F)$, we call the upper bound of the set of exponents $\eta \in \mathbb{R}$ which meet the following *Łojasiewicz inequality at infinity*:

$$|F(x)| \geqslant C|x|^{\eta} \quad \text{for } |x| > R \text{ and some constants } C > 0 \text{ i } R > 0.$$

If F is a continuous semialgebraic mapping, then $\mathcal{L}_\infty(F) \in \mathbb{Q} \cup \{-\infty\}$ and $\mathcal{L}_\infty(F) > -\infty$ if and only if the set $F^{-1}(0)$ is compact. This exponent has been considered by many authors in the context of effective Nullstellensatz in the real case and in the complex one, and for considerations of properness of mappings and of bifurcation values of polynomials in the real and complex cases (see e.g. [Brow, Ch, CK$_3$, CK$_5$, CK$_7$, Cy$_3$, CyKT, 18, 19, Ha, J$_4$, J$_5$, JK1, JKS, Ko$_1$, Ko$_2$, Ko$_3$, Pa$_2$, Pł$_1$, Sk, 26, 28, Shif, Lo$_1$, Lo$_2$, Te]). The deepest result in this direction is the Chądzyński–Kollár inequality (see [Ch, Ko$_1$, Ko$_2$, Ko$_3$, CyKT, J$_5$]):

For any polynomial mapping $F = (f_1, \dots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ with[2] *$\#F^{-1}(0) < \infty$, holds*

$$(1) \qquad \mathcal{L}_\infty(F) \geqslant d_m - B(n; d_1, \dots, d_m),$$

[1] By $F : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, we denote a mapping defined in a neighbourhood $U \subset \mathbb{R}^n$ of the point a with values in $\mathbb{R}^m$ such that $F(a) = b$.

[2] $\#A$ denotes the cardiality of a set A.

where $d_j = \deg f_j$ *for* $j = 1, \dots, m$, $d_1 \geqslant \dots \geqslant d_m > 0$ *and*

$$B(n; d_1, \dots, d_m) = \begin{cases} d_1 \cdots d_m & \text{if } m \leqslant n, \\ d_1 \cdots d_{n-1} d_m & \text{if } m > n. \end{cases}$$

A central place in my research is taken by searching quantitative and effective methods regarding the calculation of Łojasiewicz exponents (papers [11, 12, 13, 15, 16, 18, 19, 23] before the habilitation and [26, 27, 30, 39, 45, 47, 48] after the habilitation), estimating the exponents (papers [35, 38] after the habilitation) and their application to classification of singularities and characterization of bifurcation points of polynomials (papers [28, 29, 40] after the habilitation). I consider the results concering an application of the Łojasiewicz exponent for minimizing of polynomials on semialgebraic sets the most important results of my scientific achievements (papers [37] and also [33, 36] after the habilitation).

Scientific achievements before the habilitation

Before the habilitation I conducted research in the field of complex analytic and algebraic geometry. It can be divided into the following thematic groups:

1. ***Łojasiewicz exponent***
2. ***Factorization of polynomials***
3. ***Automorphisms and the Jacobian conjecture***

1. Łojasiewicz exponent. ([8, 11, 13, 18, 21, 23] and [12, 15, 16, 19] included in the habilitation dissertation). These papers are in the range of research of the Łojasiewicz exponent at a point of holomorphic and subanalytic mappings and the Łojasiewicz exponent at infinity of polynomial mappings.

a) Łojasiewicz exponent at a point. In the paper [12] the research of the Łojasiewicz exponent and multiplicity of holomorphic mappings $f = (f_1, ..., f_m) : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$, where $m > n$, at an isolated zero in $0 \in \mathbb{C}^n$, to the well researched case where $m = n$ was reduced. By $i_0(f)$ we denote the multiplicity of a mapping f at the point $0 \in \mathbb{C}^n$, i.e., isolated imptoper intersection multiplicity of the graph of f and the space $\mathbb{C}^n \times \{0\}$ at the point $(0, 0)$ in the sense of R. Achilles, P. Tworzewski and T. Winiarski [ATW] or, in the other terms – Hilberta-Samuela multiplicity of the ideal generated by components of f. If $m = n$, then $i_0(f)$ is equal to the codimension of the ideal $(f_1, ..., f_n)\mathcal{O}$, where $\mathcal{O}$ denotes the ring of holomorphic functions germs at $0 \in \mathbb{C}^n$. Namely, for a holomorphic mapping f with isolated zero at zero, we have

Theorem 1 ([12], Theorems 1.1 i 1.2). *For the generic*[3] *linear mapping* $L : \mathbb{C}^m \to \mathbb{C}^n$*, hold* $\mathcal{L}_0(f) = \mathcal{L}_0(L \circ f)$ *and* $i_0(f) = i_0(L \circ f)$.

After my habilitation the above theorem has been generalized for the Łojasiewicz exponent to the case of real mappings [30] and applied in [38]. Theorem 1 allows to transfer a number of properties of the Łojasiewicz exponent and multiplicity from the case $m = n$, to the case when

[3] i.e., for any except at most proper algebraic subset of a given set.

$m > n$, for instance: $k_n \leqslant \mathcal{L}_0(f) \leqslant i_0(f) + k_n - \prod_{i=1}^n k_i$, where $k_j = \operatorname{ord} f_j$, $k_1 \leqslant \ldots \leqslant k_m$ ([Ch] for $m = n = 2$, [Pł$_4$] for $m = n$). It also reduces considerations of the multiplicity and the Łojasiewicz exponent of holomorphic mappings to the case of polynomial mappings, namely we have

Theorem 2 ([Pł$_4$] for $m = n$, [12] for $m > n$) *If $g : \mathbb{C}^n \to \mathbb{C}^m$ is a polynomial mapping such that* $\operatorname{ord}(f - g) > \mathcal{L}_0(f)$, *then $\mathcal{L}_0(f) = \mathcal{L}_0(g)$ and $i_0(f) = i_0(g)$.*

In the erratum [12a] to the paper [12], an error was rectified in formulation of [12, Proposition 1.2].

In the paper [19] the problem of achieving of the Łojasiewicz exponent on an analytic curve was solved. This involves the following J. Bochnak i J.J. Risler theorem [BR], which was proved by means of Curve Selection Lemma:

If $X, Y \subset \mathbb{R}^n$ are closed subanalytic sets, then for any relatively compact neighbourhood $\Omega \subset \mathbb{R}^n$ of $x_0 \in X \cap Y$, the number

$$\mathcal{L}_\Omega(X, Y) := \inf\{\nu \in \mathbb{R} : \exists_{C>0} \forall_{x\in\Omega} \quad \operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geqslant C \operatorname{dist}(x, X \cap Y)^\nu\}$$

is rational.

It is known that $\mathcal{L}_{x_0}(X, Y) = \inf\{\mathcal{L}_\Omega(X, Y) : \Omega$ – relatively compact neighbourhood of $x_0\}$. If $x_0 \notin \operatorname{Int} X \cap Y$, then for any relatively compact neighbourhood Ω of x_0, by Curve Selection Lemma, the exponent $\mathcal{L}_\Omega(X, Y)$ is attained on some analytic curve $\varphi : [0, r) \to \Omega$ such that $\varphi((0, r)) \subset \Omega \setminus (X \cap Y)$ and $\varphi(0) \in X \cap Y$, i.e. the following holds,

$$\operatorname{dist}(\varphi(t), X) + \operatorname{dist}(\varphi(t), Y) \leqslant C' \operatorname{dist}(\varphi(t), X \cap Y)^\nu \quad \text{for} \quad t \in [0, r), \tag{2}$$

where $\nu = \mathcal{L}_\Omega(X, Y)$ and $C' > 0$ is a constant. The problem of achieving the Łojasiewicz exponent relates to the following question: *is there a curve φ satisfying* (2) *with $\varphi(0) = x_0$?* The answer to this question is negative, which we have shown in [19, Example 2.5]. This fact was the main barrier to show that $\mathcal{L}_{x_0}(X, Y) \in \mathbb{Q}$. This difficulty has now been overcome in the paper [19] thanks to the Lipschitz stratifications introduced by T. Mostowski [Mos], which admits local bi-Lipschitz trivialization of subanalytic sets (which was proved by A. Parusiński [Pa$_1$]). Namely, we have

Theorem 3 ([19], Theorems 1.3, 1.4) *Let $X, Y \subset \mathbb{R}^n$ be closed subanalytic sets, and let $x_0 \in X \cap Y$.*

(i) *Then $\mathcal{L}_{x_0}(X, Y) \in \mathbb{Q}$ and the inequality* (S) *holds with $\nu = \mathcal{L}_{x_0}(X, Y)$, some neighbourhood Ω of x_0 and $C > 0$.*

(ii) *If $x_0 \notin \operatorname{Int} X \cap Y$, then $\mathcal{L}_{x_0}(X, Y)$ is attained on some analytic curve, i.e.,* (2) *holds with $\nu = \mathcal{L}_{x_0}(X, Y)$.*

The proof of Theorem 3 was carried out in a slightly more general situation (see [19, Theorem 1.5]). Namely, for three closed subanalytic sets X, Y, Z such that $X \cap Y \subset Z$ and a point $x_0 \in X \cap Y$ we showed that the smallest exponent $\nu \in \mathbb{R}$ satisfying the inequality

$$\operatorname{dist}(x, Y) \geqslant C \operatorname{dist}(x, Z)^\nu \qquad \text{for} \qquad x \in \Omega \cap X$$

and some $C > 0$ and a neighbourhood Ω of x_0, is a rational number and (under the assumption that $x_0 \notin \operatorname{Int} X \cap Y$) this exponent is attained on an analytic curve. The smallest exponent is denoted by $\mathcal{L}_{x_0}(X; Y, Z)$. A Lipschitz stratification $X \cup Y \cup Z = \bigcup_{\alpha \in A} S_\alpha$ preserves the Łojasiewicz exponent, i.e., the function $X \cap Y \ni x \mapsto \mathcal{L}_x(X; Y, Z)$ is constant on each stratum $S_\alpha \subset X \cap Y$ ([19, Corollaries 2.6, 2.7]). In particular, the sert $\{\mathcal{L}_x(X; Y, Z) : x \in X \cap Y\}$ is finite. The main reason for considerations of the exponent $\mathcal{L}_{x_0}(X; Y, Z)$ insetad of $\mathcal{L}_{x_0}(X, Y)$ was the following observation ([19, Corollary 3.1]):

If $X \subset \mathbb{R}^n$ is a closed subanalytic set, then for any continuous subanalytic mapping $f : (X, x_0) \to (\mathbb{R}^m, 0)$,

$$\mathcal{L}_{x_0}(f) = \mathcal{L}_{(x_0,0)}(\Gamma(f); X \times \{0\}, V \times \mathbb{R}^m),$$

where $\Gamma(f)$ is the graph of f and $V = f^{-1}(0)$.

It follows that for a continuous semialgebraic mapping $f : X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$ is a closed set, there is a universal exponent $\mathcal{L}(f) \in \mathbb{Q}$ such that for $\nu = \mathcal{L}(f)$, the Łojasiewicz inequality $|f(x)| \geqslant C \operatorname{dist}(x, V)^\nu$ takes place in a neighborhood of every point of the set V. This allows us to transfer S. Ji, J. Kollár and B. Shiffmana inequality [JKS] to the case of semialgebraic mappings ([19, Theorem 3.5]):

$$\text{(JKS)} \qquad |f(x)|(1 + |x|)^l \geqslant C \operatorname{dist}(x, V)^{\mathcal{L}(f)} \quad \text{on} \quad X$$

for some $l \in \mathbb{N}$ and $C > 0$. The inequality (JKS) and the exponent $\mathcal{L}_{x_0}(f)$ also transfer for two mappings ([L-JT], [19, Corollary 4.1, Theorem 4.5]).

b) Łojasiewicz exponent at infinity. Let $f = (f_1, ..., f_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping. In the paper [21] considerations of fibers od f and the exponent $\mathcal{L}_\infty(f)$ were reduced to the well studied case, when $m = n$ ([Ch, Pł$_3$]). Namely, it was showin that

Theorem 4 ([21], Theorem 1, Corollary 1) *If $m > n > 1$, then there exists a polynomial mapping $g : \mathbb{C}^n \to \mathbb{C}^n$ such that the fibre $f^{-1}(0)$ is a union of some irreducible components of the fibre $g^{-1}(0)$. If additionally $f^{-1}(0)$ is a finite set, then $g^{-1}(0)$ is finite, too, and $\mathcal{L}_\infty(f) \geqslant \mathcal{L}_\infty(g)$.*

The above theorem was used in the paper [CyKT] to reduction of the proof of a Chądzyński-Kollar type inequality (see (1)) to the case of polynomial mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$.

In the paper [16] the research from [21] was continued. Similarly as in [12], the research of $\mathcal{L}_\infty(f)$ for a polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^m$, with $m > n$, was reduced to the case, when $m = n$, by composition of the mapping with the generic linear mapping (see [16, Theorem 2.1]).

The exponent $\mathcal{L}_\infty(f)$ characterizes the properness of a polynomial mapping f (see [Ch, Pł$_3$]), i.e., the mapping f is proper if and only if $\mathcal{L}_\infty(f) > 0$. Based on this property, in the paper [11] joint with T. Rodak, is shown:

Theorem 5 ([11]) *Any polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ which is closed in the Zariski topology is also proper.*

In the paper [13], joint with T. Krasiński, the above theorem was generalized to the case of complex regular mappings. This result was generalized by Z. Jelonek [J$_2$] to the case of regular mappings over arbitrary algebraic closed fields.

For considerations of bifurcation points [4] of a polynomial $f:\mathbb{C}^n\to\mathbb{C}$, we use the following global Bochnak and Łojasiewicz inequality [BŁ] and the gradient inequality, which were both proved in [15] and in the paper [18] joint with J. Gwoździewicz:

Theorem 6 ([15], Theorem 1 and [18], Theorem 3.1) *There exist constants $C, R, \varepsilon > 0$ such that*

$$|f(z)|\geqslant R \;\Rightarrow\; |z||\nabla f(z)|\geqslant C|f(z)|,$$
$$|f(z)|\leqslant \varepsilon \;\Rightarrow\; |z||\nabla f(z)|\geqslant C|f(z)|.$$

For a polynomial in two variables $f:\mathbb{C}^2\to\mathbb{C}$ we proved a global version of the Łojasiewicz gradient inequality [$Ł_4$] (see [15, Corollary 4] and [18, Theorem 7.5]):

If $d=\deg f>0$, then there exist constants $C, R, \varepsilon>0$ such that

$$|f(z)|\geqslant R \;\Rightarrow\; |\nabla f(z)|\geqslant C|f(z)|^{-(d-1)^2},$$
$$|f(z)|\leqslant \varepsilon \;\Rightarrow\; |\nabla f(z)|\geqslant C|f(z)|^{(d-1)^2}.$$

In the multidimensional case the above inequalities hold under the additional assumption that f satisfies the so-called Fedorjuk condition.

The set of bifurcation values at infinity of a polynomial f we denote by $B_\infty(f)$. Studies of this set can be found for example in [CK_8], [GwP], [Ha], [Pa_2], [Ve]. From the above inequalities there follows a known fact that *the set $B_\infty(f)$ is finite* and

Theorem 7 ([18], Corollary 6.1, cf. [CK_4, GwP, Ha, 15]) *If $d=\deg f>2$, then*

$$\lambda\in B_\infty(f) \;\Leftrightarrow\; \mathcal{L}_\infty(f-\lambda,\, \operatorname{grad} f)\leqslant -\frac{1}{d-2}. \tag{3}$$

J. Chądzyński and T. Krasiński [CK_4] showed that the estimate $\mathcal{L}_\infty(f-\lambda,\, \operatorname{grad} f)\leqslant -\frac{1}{d-2}$ in (3) cannot be improved. For considerations in the paper [18] very useful was the following theorem of Charzyński, Kozlowski and Smale type. It is a consequence of the Koebe Quarter Theorem (cf. [23] and [CK_4, Proposition 2.3]).

Theorem 8 ([18], Theorem 2.1) *Let $P:\mathbb{C}\to\mathbb{C}$ be a polynomial of degree $d>1$, $\varphi_1,...,\varphi_d\in\mathbb{C}$ and $\xi_1,...,\xi_{d-1}\in\mathbb{C}$ be respectively, all the roots of P and the derivative P'. Then*

$$\min_{1\leqslant k\leqslant d-1}|P(\xi_k)| \;\leqslant\; 4\min_{i\neq j}(|\varphi_i-\varphi_j||P'(\varphi_i)|).$$

In the paper [23], based on Theorem 8, are given new elementary proofs of known formulas of T. C. Kuo Y. C. Lu [KLu] for the Łojasiewicz exponent of the gradient of a holomorphic function at zero and of H. V. Ha [Ha] for the Łojasiewicz exponent at infinity of the gradient of a polynomial.

The paper [8], joint with T. Krasiński, is devoted to the comparison of complex and real bifurcation sets of a polynomial $f:\mathbb{R}^2\to\mathbb{R}$.

[4] We say that a point $\lambda\in\mathbb{C}$ is *typical at infinity* for a function $f:\mathbb{C}^n\to\mathbb{C}$, if there exist a neighbourhood $U\subset\mathbb{C}$ of λ and a compact set $K\subset\mathbb{C}^n$ such that $f: f^{-1}(U)\setminus K\to U$ is a C^∞ trivial fibration. Othervise, we call the point λ *bifurcation at infinity*. Analogously we define bifurcation points at infinity for real functions.

2. Factorization of polynomials. ([1, 5, 7, 9, 10, 17]). These papers are devoted to the issues of reducibility of polynomials and the way of presenting them in a "simpler" form. Similar studies can be found in the following articles [BY, Cy_1, Fur, Nett, $\mathrm{Pł}_5$, $\mathrm{Pł}_6$, Sa] and in the monograph [Schin].

In the paper [5] was given a description of multivalued algebraic mappings, that is, sets of germs of analytic mappings related by analytic extension along curves, whose coordinates satisfy algebraic equations. These results were used to construct the field of Nash functions [7] and to show that this is an algebraic closure of the rational functions field of several variables. The main difficulty in this construction was the choice of appropriate filtering of the space $\mathbb{C}^n$ by a family of simply connected domains $\Omega_P \subset \mathbb{C}^n$ indexed by polynomials $P \in \mathbb{C}[x]$, $x = (x_1, ..., x_n)$ such that $P(z) \neq 0$ for $z \in \Omega_P$ and closed with respect to intersection, i.e., $\Omega_{PQ} = \Omega_P \cap \Omega_Q$. In [7] were given some conditions for irreducibility of polynomials with coefficients in the field of Nash function. These studies were continued in the paper [32] after the habilitation.

The articles [9] and [10], joint with M. Frontczak and P. Skibiński, are devoted to theorems of Bertini-Krull type on reducibility of polynomials (see [Cy_1, Fur, $\mathrm{Pł}_5$, Sa, Schin]). Let R be the ring of holomorphic functions in a domain $G \subset \mathbb{C}^m$ or $R = \mathbb{K}[\lambda]$, $\lambda = (\lambda_1, ..., \lambda_m)$, where $\mathbb{K}$ is an algebraic closed field of characteristic zero. Main results of the paper [10] concern some conditions which admit a representation of a polynomial $F \in R[x]$ in the form $h(\lambda, Q(\lambda, x))$, where $h \in R[z]$, $Q \in R[x]$ and $\deg Q < \deg P$. For instance, such a condition is: reducibility of the polynomial $P(\lambda, x) - \tau \in \mathbb{C}[x]$ with fixed parameters λ, τ ([10, Theorem 1, Corollary 6], see also [Cy_1, Fur, $\mathrm{Pł}_5$, Schin]). A continuation of [10] is the publication [9] which concerns Solomon theorem on estimation of the dimension of space spanned by the coefficients of the factors in the decomposition of a polynomial. The main result of [9] can be written as follows:

Theorem 9 ([9], Theorem 2) *Let $\mathbb{K}$ be an algebraic closed field and let $\lambda = (\lambda_1, ..., \lambda_m)$, $x = (x_1, ..., x_n)$ be systems of variables. If a polynomial $F \in \mathbb{K}[\lambda, x]$ is irreducible over $\mathbb{K}(\lambda)$, then the number of linearly independent over $\mathbb{K}$ coefficients of any irreducible factor of F over the separable closure of the field $\mathbb{K}(\lambda)$ does not exceed the number of integer points of the Newton polyhedron of the polynomial F.*

In the case $m = 1$, this result gives the Solomon theorem (see [Sa, Schin]).

The paper [17], joint with A. Nowicki, is devoted to the decomposition of a polynomial into so-called imaginary parts. Namely, let $L = k[\xi]$ be a finite extension of a field k of characteristic zero and let $\varphi(t) = t^m - a_{m-1}t^{m-1} - \cdots - a_0$ be the minimal polynomial for ξ. By the *imaginary decomposition* of a polynomial $f \in L[z]$ in a single variable z we call the following decomposition to a sum:

$$f(x_0 + \xi x_1 + \cdots + \xi^{m-1}x_{m-1}) = u_0 + \xi u_1 + \cdots + \xi^{m-1}u_{m-1},$$

where $u_0, ..., u_{m-1} \in k[x]$, $x = (x_0, ..., x_{m-1})$. The sequence $u = (u_0, ..., u_{m-1})$ we call a sequence of *imaginary parts* of the polynomial f. The main result of this paper is the following

Theorem 10 ([17], Theorem 3.8) *A sequence $u = (u_0, ..., u_{m-1})$ is a sequence of imaginary parts of some polynomial $f \in L[z]$ if and only if u satisfies the following generalized Cauchy-Riemann conditions:*

$$\frac{\partial u}{\partial x_i} = \frac{\partial \overline{u}}{\partial x_{i-1}}, \qquad i = 1, ..., m-1,$$

where $\overline{u} = (\overline{u}_0, ..., \overline{u}_{m-1})$ *is defined by* $\overline{u}_i = a_i u_{m-1} + u_{i-1}$, $u_{-1} = 0$.

Moreover we proved that the imaginary parts of a polynomial f are coprimes in $k[x]$. In the case $m = 2$, we also gave their application in the number theory.

In the paper [1] we gave a new elementary proof of the following theorem:

Theorem 11 ([1]) *The Jacobian of a homogeneous polynomial mapping* $F = (F_1, ..., F_n) : \mathbb{K}^n \to \mathbb{K}^n$, *where* $\mathbb{K}$ *is a field of characteristic zero, having a non-trivial zero belongs to the ideal generated by* $F_1, ..., F_n$.

The above theorem is known from the E. Netto book [Nett, p. 142], however, the proof did not meet the current requirements of precision. It has great importance in estimating of the Noether exponent (see [Pł$_6$]). It is also true for holomorphic mappings (see [BY]).

A part of results from the paper [14] can also be included for this subject. Due to the relationships with automorphisms it was placed in the thematic group 3.

3. Automorphisms and the Jakobian conjecture. ([2, 3, 4, 14, 24]). These articles are in the circle of studies on the Jacobian conjecture which says that a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ with a constant nonzero Jacobiam $J(F)$ is a $\mathbb{C}$ automorphism of the ring $\mathbb{C}[x]$, $x = (x_1, ..., x_n)$. This issue is dealt with in many papers, for example [BCW, CCS, Dr, E, Ka, Ru, RW, St, Wr].

The main theorem in the paper [2] says that the composition process of the following Whitney operators

$$W_i(M) = J(M, F_1, ..., F_{i-1}, F_{i+1}, ..., F_n), \quad M \in \mathbb{C}[x], \quad i = 1, ..., n,$$

is commutative if and only if $J(F) = \text{const}$. This result is a generalization of a result by Z. Charzyński, J. Chądzyński and P. Skibińskiego in [CCS]. The operators W_i were used in the paper [4], joint with T. Krasinski, to give a criterion for a polynomial mapping to be an automorphism of the ring $\mathbb{C}[x]$. This criterion is a transfer of two-dimensional result by Y. Stein [St] to the multi-dimensional case.

The main result of the paper [14], joint with T. Krasiński, is the following

Theorem 12 ([14], Theorem 3) *For each open polynomial mapping* $F : \mathbb{C}^n \to \mathbb{C}^m$, *where* $m \geqslant 3$, *there exists an nonsingular linear change of coordinates* $\alpha : \mathbb{C}^m \to \mathbb{C}^m$ *such that any component of the mapping* $\alpha \circ F$ *is totally primitive, ie. its each fiber is irreducible.*

The above result is a multidimensional generalization of a result by S. Kaliman [Ka] for $m = n = 2$, under additional assumption $J(F) = 1$. The method of the proof of Theorem 12 does not include the two-dimensional case.

It is known that each $\mathbb{C}$-automorphisms of the ring $\mathbb{C}[x, y]$ is tame, i.e., it is the composition of finite number of linear and triangular $\mathbb{C}$-automorphisms. In 2003., I. P. Shestakov and U. U. Umirbaev [ShU] proved that it is not true for the Nagat automorphism [Na] of the ring $\mathbb{C}[x, y, z]$. In the paper [24] (and [20] published after the habilitation) it is shown that the Nagata automorphism considered as a $\mathbb{C}$-automorphism of the ring $\mathbb{K}[x, y, z, w]$ is tame, where $\mathbb{K}$ is an arbitrary ring with unity. This result in 1989. was proved by M. K. Smith [Sm] on the assumption that $\mathbb{K}$ is a ring of characteristic zero.

Scientific achievements after the habilitation

My scientific research after the habilitation are located in real and complex analytic and algebraic geometry. Recently they focus on the real algebraic geometry. They can be divided into the following thematic groups:

1. **Effective methods for calculation of the Łojasiewicz exponent**
2. **Effective methods for estimating of the Łojasiewicz exponent**
3. **Topologic types of mappings**
4. **Sum of squares and optimization**
5. **Real fields**

An important place in my research is occupied by searching effective and quantitative methods regarding calculations of the Łojasiewicz exponents (papers [26, 27, 30, 39, 45, 47, 48]). In the papers [26, 27, 30, 39] we use methods of algebraic and analytic geometry and the improper intersection theory. The papers [45, 47, 48] are review articles.

Amongst achievements after the habilitation, connected with the Łojasiewicz exponent, an important place is taken by its applications in the classification of singularities (papers [28, 29, 40]) relationships of the Łojasiewicz exponent of the gradient of a polynomial near its fibre, with the trivialization of this polynomial in a neighbourhood of this fibre [28]; impact of the Łojasiewicz exponent at infinity of the Rabier function of a mapping differential for the isotopicallity of this mapping in a neighbourhood of infinity [29]; impact of the Łojasiewicz exponent of the gradient of a function with non isolated singularity to a finite determinability of the jet of this function [40]. In above papers an important role was played by methods of differential geometry, differential equations about properties of flows of vector fields, computational algebra and algebraic geometry, including Lipschitz stratification and properties of the set of points at which a mapping is not proper.

As the most important of my results after the habilitation I regard an application of the Łojasiewicz exponent to minimize of a polynomial on a compact semialgebraic set, and provide a method for the approximate determination of the critical points of a polynomial on a convex semialgebraic set (paper [37], by using the papers [35, 38] about estimating of the Łojasiewicz exponents). To these problems are related: the paper [33] about the impact of bifurcation points of the polynomial f for the stability of the algebras of bounded polynomials on the sets $f^{-1}((-\infty, a])$; effective version of M. Putinar and F.-H. Vasilescu theorem [PV$_2$] (see [36]) and papers [31, 34] on extension a regular mappings to polynomial mappings with preserving the Łojasiewicz exponent, and expansion to a sum of squares of polynomials. To obtain the above results, in [35] we used the methods of differential equations in the range of gradient field of flows [KMP], differential geometry, algebraic and semialgebraic geometry and convex analysis.

At the end of the scientific report we will present an universal geometric model of the real closed field. In particular, we will give a characterization of Archimedean fields in terms of this model [32]. It allows us to interprete of derivations in real fields. In the above paper we used methods of semialgebraic geometry in the range of real fields, including the results of A. Tarski [Ta1, Ta2].

1. Effective methods for calculation of the Łojasiewicz exponent. This subject of research is devoted to finding effective methods of calculating the Łojasiewicz exponent: of holomorphic mappings in an isolated zero, and of polynomial mapping at infinity. These studies are a continuation of research initiated by J. Chądzyński and T. Krasiński [CK_1, CK_2, CK_3, CK_4, CK_5, CK_6, CK_7, CK_8].

a) ***Łojasiewicz exopnent at a point.*** To this cubject are devoted the papers [27], joint with T. Rodak, [39], joint with T. Rodak and A. Różycki and partially – [30, 35], which we will discuss later.

In the paper [27] we give an effective formula for the Łojasiewicz exponent of a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^m$, $m \geqslant n$, with isolated zero in the point $0 \in \mathbb{C}^n$. Take mappings

$$\begin{aligned} &H_{L,M}(z) = L(F(z), M(z)) + (z_1^{d^n+1}, \dots, z_n^{d^n+1}),\\ &\Phi_q(L, M, N, z) = \big(L, M, N, H_{L,M}(z), N(z)\big), \end{aligned}$$

where $d = \deg F$ and $M \in \mathbb{L}(n, q)$, $L \in \mathbb{L}(m+q, n)$, $N \in \mathbb{L}(n, 1)$, $q \in \{0, \dots, n\}$, and $\mathbb{L}(m, n)$ denotes the set of all linear mappings $\mathbb{C}^m \to \mathbb{C}^n$. Using methods of computational algebra and algebraic geometry, we can effectively calculate a polynomial $P_q \in \mathbb{C}[L, M, N, y, t]$ of the form

$$P_q(L, M, N, y, t) = \sum_{j=0}^{p} P_{q,j}(L, M, N, y)t^j,$$

such that $P_{q,p} \neq 0$, which describes the set of values of the mapping Φ_q, where, in the above formula, L, M, N denote suitable matrices of variables. The main result of the paper [27] is the following theorem

Theorem 13 ([27], Theorem 7) *Let $F(0) = 0$ and let $V = F^{-1}(0)$. Then there exists r, $0 \leqslant r < p$ such that $\operatorname{ord}_y P_{q,j} > 0$ for $j = 0, \dots, r$ and $\operatorname{ord}_y P_{q,r+1} = 0$. Put*

$$\Delta'(P_q) = \min_{j=0}^{r} \frac{\operatorname{ord}_y P_{q,j}}{r + 1 - j}.$$

1. *If $\dim_0 V \geqslant q + 1$, then $1/\Delta'(P_q) \geqslant d^n + 1$.*
2. *If $\dim_0 V < q + 1$, then $\mathcal{L}_0(F, M) = 1/\Delta'(P_q) < d^n + 1$.*

The above theorem is a generalization of a result by A. Płoskiego [$\mathrm{Pł}_4$] for mappings with isolated zeroes at zero such that $m = n$ and a result by J. Chądzyński and T, Krasiński [CK_4] for $n = m = 2$. From Theorem 13 there also follows that (see [27, Corolary 8]),

$$\dim_0 V = \min \left\{ q : 1/\Delta'(P_q) < d^n + 1 \right\},$$

i.e., we get an effective formula for the dimension of the germ at zero of an algebraic set.

The paper [39] is devoted to studying of properties of the Łojasiewicz exponent for a holomorphic deformation of a holomorphic mapping. In this paper we show that for a holomorphic deformation $(f_s)_{s\in S}$ of a function f, there exists a stratification of the set of parameters S such that the funkction $s \mapsto \mathcal{L}_0(f_s)$ is constant on each stratum of this stratification. Under notations of Theorem 13 we obtain the following effective formula for multiplicity $i_0(F)$ of the mapping F at an isolated zero.

Theorem 14 ([39], Theorem 4) *If the mapping F has an isolated zero at zero, then*

$$i_0(F) = \min\{j \in \mathbb{Z} : \operatorname{ord}_y P_{0,j} = 0\}.$$

b) *Łojasiewicz exponent at infinity.* To this exponent are devoted the papers [26], joint with T. Rodak, [30], joint with A. Szlachcińska, and partially - the papers [28, 29, 35], which we will discuss further.

In the paper [26] we give an effective procedure for calculating of the Łojasiewicz exponent of a polynomial mapping at infinity. At first we give a formula for the exponent in the case of proper mappings [26, Theorem 2] (in similar terms as in the paper [27]), and then using the Chądzyński-Kollár inequality (1), we reduce counting of this exponent for any mappings to the case of proper mappings.

In the proofs of Theorems 13 and [26, Theorem 2] was used a method of calculating the Łojasiewicz exponent at a point and at infinity of an overdetermined mapping, ie. $\mathbb{C}^n \to \mathbb{C}^m$, when $m > n$, by reducing the considerations to the case when $m = n$, as a result of the composition of the mapping with the generic linear mapping $\mathbb{C}^m \to \mathbb{C}^n$ (based on the articles [16, 12] obtained before the habilitation). This method was generalized to the case of real mappings in the paper [30].

2. Effective methods for estimating of the Łojasiewicz exponent. The search for estimates of Łojasiewicz exponents at a point and at infinity are related to the classification problems of local and global singularities and with optimization problems, which we describe later. In the papers [35], joint with K. Kurdyka, and [38], joint with K. Kurdyka and A. Szlachcińska, we obtain effective estimates of these exponents for polynomial mappings, as well as regular and semialgebraic mappings. In particular, we give a generalization of Chądzyński-Kollár inequality (1). We give also estimates of the Łojasiewicz exponent for local and global regular separation of algebraic and semialgebraic sets. A starting point in these considerations is an effective estimate of the Łojasiewicz exponent in the gradient inequality for polynomials, by D. D'Acunto and K. Kurdyka [AK$_1$, AK$_2$] and independently by A. Gabrielov [Ga$_2$]. In the case of mappings with an insulated zero, effective estimate in this regard was given by J. Gwoździewicz [Gw] and J. Kollár [Ko$_2$].

a) *Łojasiewicz gradient inequality.* For an analytic function $f : (\mathbb{R}^n, a) \to (\mathbb{R}, 0)$ with an isolated zero at the point a, J. Gwoździewicz [Gw] (cf. [Te] for complex functions and [Ph$_1$] for subanalytic functions) proved that there are the following relations of the Łojasiewicz exponent in the gradient inequality $\varrho_a(f)$ and the Łojasiewicz exponents $\mathcal{L}_a(f)$ of a function f and its gradient ∇f:

$$\text{(G1)} \qquad \mathcal{L}_a(f) = \frac{1}{1 - \varrho_a(f)} = \mathcal{L}_a(\nabla f) + 1.$$

This result is not true in the general case, even if we assume that f has an isolated singularity at the point a.

In the paper [35], without any assumptions about the set of zeros of the function f, we show that the following inequality holds (see [35, Corollary 1])

$$\text{(4)} \qquad \mathcal{L}_a(f) \leqslant \frac{1}{1 - \varrho_a(f)}.$$

If additionally f has an isolated singularity at the point a, then (see [35, Corollary 3])

$$\frac{1}{1-\varrho_a(f)} \leqslant \mathcal{L}_a(\nabla f)+1. \tag{5}$$

From (G1) we see that the estimates (4) and (5) can not be improved in terms of of the exponent $\varrho_a(f)$.

We prove the above results by technique of differential equations and more precisely by technique of gradient flows. They arise from the following theorem [35, Theorem 1] (see. [KMP]):

Assume that the function f satisfies the inequality (Ł$_1$) *in a neighbourhood U of the point a and let $\gamma : [0, s) \to U \setminus V$ be the global solution to the right of the differential equation $x' = H(x)$, where*

$$H(x) = -\operatorname{sign} f(x) \frac{\nabla f(x)}{|\nabla f(x)|} \quad \textit{for } x \in U \setminus V$$

and $V = V(f)$. If $\gamma(0)$ is sufficiently close to the point a, then

$$\operatorname{dist}(\gamma(0), V(f)) \leqslant \operatorname{length}\gamma \leqslant \frac{1}{(1-\varrho)C}|f(\gamma(0))|^{1-\varrho}.$$

By the above and by D. D'Acunto and K. Kurdyka inequality [AK$_2$] (or by A. Gabrielov inequality [Ga$_2$])

$$\varrho_a(f) \leqslant 1 - \frac{1}{d(3d-3)^{n-1}} \tag{D-K}$$

for any polynomial $f \in \mathbb{R}[x]$ of degree $d \geqslant 2$ such that $f(a) = 0$, we obtain: $\mathcal{L}_a(f) \leqslant d(3d-3)^{n-1}$ (see [35, Corollary 5]) and $\mathcal{L}_a(\nabla f) \leqslant (d-1)(6d-9)^{n-1}$ (see [35, Remark 4]). For a polynomial mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ of degree d, we have

$$\mathcal{L}_a(F) \leqslant d(6d-3)^{n-1}$$

(see [35, Corollary 6]). In the case of polynomial f with an isolated zero at the point a, J. Gwoździewicz [Gw] proved that $\mathcal{L}_a(f) \leqslant (d-1)^n + 1$.

b) *Regular separation of algebraic and semialgebraic sets.* For complex algebraic sets $X, Y \subset \mathbb{C}^n$ (of pure dimensions) and a point $x_0 \in X \cap Y$, as the exponent ν in the inequality (S), can be inserted $\deg_{x_0} X \cdot \deg_{x_0} Y$. It is a particular case of results by E. Cygan [Cy$_3$], E. Cygan, T. Krasiński i P. Tworzewski [CyKT] oraz S. Ji, J. Kollár i B. Shiffman [JKS].

In the paper [35] we transfer these results to the case of real algebraic sets and regular mappings and in the paper [38] – to the case of semialgebraic sets and mappings. In the case of real algebraic sets $X, Y \subset \mathbb{R}^n$ (respectively regular mappings $F|_X$), for such estimates it suffices to take into account degrees of polynomials describing sets $X, Y \subset \mathbb{R}^n$ (respectively degrees of polynomials describing the domain X of the mapping and the degree $\deg F$). In the case of semialgebraic sets and mappings we use for this purpose two parameters, described below, characterizing the complexity of semialgebraic sets and functions.

Let $X \subset \mathbb{R}^n$ be a closed semialgebraic set. Then there existsa decomposition

$$X = X_1 \cup \cdots \cup X_k \tag{6}$$

into union of basic closed semialgebraic sets $X_1, \dots, X_k$ of the form

$$X_i = \{x \in \mathbb{R}^n : g_{i,1}(x) \geqslant 0, \dots, g_{i,r_i}(x) \geqslant 0,\ h_{i,1}(x) = \dots = h_{i,l_i}(x) = 0\}, \tag{7}$$

$i = 1, \dots k$ (see [BCRo]), where $g_{i,1}, \dots, g_{i,r_i}, h_{i,1}, \dots, h_{i,l_i} \in \mathbb{R}[x_1, \dots, x_n]$. Suppose that r_i is the smallest possible number of inequalities $g_{i,j}(x) \geqslant 0$ in definition of the set X_i for $i = 1, \dots, k$ and let $r(X)$ be the smallest of the numbers $\max\{r_1, \dots, r_k\}$ after all possible decompositions of X of the form (6) into unions of sets of the form (7). L. Bröcker [Bröc$_3$] (por. [Bröc$_2$, Sche$_1$]) proved that

$$r(X) \leqslant n(n+1)/2. \tag{8}$$

Denote by $\kappa(X)$, the smallest of the numbers $\max\{\deg g_{1,1}, \dots, \deg g_{k,r_k}, \deg h_{1,1}, \dots, \deg h_{k,l_k}\}$, after all possible decompositions of X of the form (6) into unions of sets of the form (7), provided $r_i \leqslant r(X)$ for $i = 1, \dots, k$.

The numbers $r(X)$ and $\kappa(X)$ characterize the complexity of the set X (see. e.g. [BPR, BCRo, RV]). Obviously $r(X) = 0$ holds if and only if the set X is algebraic.

The main result of the paper [38] for the local Łojasiewicz exponent is the following

Theorem 15 ([38], Theorem 2.1) *Let $X, Y \subset \mathbb{R}^n$ be closed semialgebraic sets. Assume that $0 \in X \cap Y$. Put $r = r(X) + r(Y)$ and $d = \max\{\kappa(X), \kappa(Y)\}$. Then there exist a neighbourhood $U \subset \mathbb{R}^n$ of the point 0 and a positive constant C such that*

$$\operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geqslant C \operatorname{dist}(x, X \cap Y)^{d(6d-3)^{n+r-1}} \quad \textit{for } x \in U. \tag{9}$$

Using results by J. Gwoździewicza in [Gw], we also get a version of theorem 15 in the case when the point 0 is an isolated point of the set $X \cap Y$ with the exponent equal to $[(2d-1)^{n+r} + 1]/2$.

In the case when X and Y are algebraic sets, the inequality (9) holds with the exponent $d(6d-3)^{n-1}$ (see [35, Corollary 8]).

From Theorem 15 we get the following estimate of the local Łojasiewicz exponent.

Theorem 16 ([38], Corollary 2.2) *Let $F : X \to \mathbb{R}^m$ be a continuous semialgebraic mapping, where the set $X \subset \mathbb{R}^n$ is closed and assume that $0 \in X$ and $F(0) = 0$. Put $r = r(X) + r(\operatorname{graph} F)$ and $d = \max\{\kappa(X), \kappa(\operatorname{graph} F)\}$. Then*

$$\mathcal{L}_0(F|X) \leqslant d(6d-3)^{n+r-1}. \tag{10}$$

The inequality (10) is crucial in estimating the "speed" of the convergence of the algorithm (based on the so-called semi-definite programming) to minimize a polynomial on a basic semialgebraic set. It enabled in [37] to reduce the problem of minimizing the polynomial on a compact semilgebraic set to minimizing the polynomial on the ball, that is to the problem much simpler (see [Schw$_2$]). We describe it in section 4.

It is known that for continuous semialgebraic functions $f, g : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is a closed semialgebraic set and $0 \in X$, fulfilling $f^{-1}(0) \subset g^{-1}(0)$, there are positive constants C, η, ε such that the following *Łojasiewicz inequality* holds (see e.g. [BCRo]):

$$|f(x)| \geqslant C|g(x)|^\eta \quad \text{if } x \in X,\ |x| < \varepsilon. \tag{11}$$

The lower bound of the exponents η in (11) is called the *Łojasiewicz exponent* of the pair (f,g) *on the set* X *at* 0 and is denoted by $\mathcal{L}_0(f,g|X)$. It is known (see [BR, 19]) that $\mathcal{L}_0(f,g|X)$ is, in the general case, a rational number; moreover, inequality (11) holds actually with $\eta = \mathcal{L}_0(f,g|X)$ for some $\varepsilon, C > 0$. Based on the effective Tarski-Seidenberg theorem [He], P. Solernó [So] presented the following asymptotic estimate for $\mathcal{L}_0(f,g|X)$:

$$(\mathrm{S}_a) \qquad \mathcal{L}_0(f,g|X) \leqslant D^{M^{ca}},$$

where D is a bound for the degrees of the polynomials involved in descriptions of f, g and X; M is the number of variables in these formulas (so in general $M \geqslant N$); a is the maximum number of alternating blocs of quantifiers in these formulas; and c is an (unspecified) universal constant.

In Theorem 16 only the function $g(x) = \operatorname{dist}(x, X \cap F^{-1}(0))$ is defined by a formula which is not quantifier-free, and it has two alternating blocs of quantifiers, hence $a = 2$. So Solernó's estimate (S_a) reads $\mathcal{L}_0^{\mathbb{R}}(F|X) \leqslant d^{(N+2)^{2c}}$, which is asymptotically comparable with our estimate (10) since $r(X) \leqslant \frac{1}{2}N(N+1)$ by (8). Needless to say, our estimate is explicit and independent from the constant c. Probably the constant c is greather than 1.

c) ***Global regular separation of algebraic and semialgebraic sets.*** In the case of complex algebraic sets $X, Y \subset \mathbb{C}^n$ of pure dimensions, E. Cygan in [Cy_3] (cf. [Brow, JKS, Ko_1, Ko_2, Ko_3]) proved the following global, called Hörmander-Łojasiewicz, inequality [Hö]:

$$(\mathrm{C}) \qquad \operatorname{dist}(z,X) + \operatorname{dist}(z,Y) \geqslant C\left(\frac{\operatorname{dist}(z, X\cap Y)}{1+|z|^2}\right)^{\deg X \cdot \deg Y} \quad \text{for} \quad z \in \mathbb{C}^n,$$

where C is a positive constant. This inequality is strongly related to the effective Nullstellensatz obtained by J. Kollár [Ko_1].

For real algebraic sets we have the following version of inequality (C).

Theorem 17 ([35], Theorem 2, Corollary 10) *If* g, $h : \mathbb{R}^n \to \mathbb{R}^m$ *are polynomial mappings,* $X = V(g)$*,* $Y = V(h)$ *and* $d = \max\{\deg g, \deg h\}$*, then for some constant* $C > 0$*,*

$$(12) \qquad \operatorname{dist}(x,X) + \operatorname{dist}(x,Y) \geqslant C\left(\frac{\operatorname{dist}(x, X\cap Y)}{1+|x|^2}\right)^{d(6d-3)^{n-1}} \quad \text{for} \quad x \in \mathbb{R}^n.$$

In particular for a polynomial mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ *of degree* d*, there exists* $C > 0$ *such that*

$$|F(x)| \geqslant C\left(\frac{\operatorname{dist}(x, V(F))}{1+|x|^2}\right)^{d(6d-3)^{n-1}} \quad \text{for} \quad x \in \mathbb{R}^n$$

If the zero-set of a polynomial mapping F is compact, Theorem 17 implies an estimete of Chądzyński-Kollár type (1) in the real case (see [35, Corollary 11]):

$$\mathcal{L}_\infty(F) \geqslant -d(6d-3)^{n-1},$$

where $d = \deg F$.

By using the E. Cygan [Cy_3] methods and inequality (12), we get the following version of inequality (C) for semialgebraic sets.

Theorem 18 ([38], Theorem 3.2) *Let $X, Y \subset \mathbb{R}^n$ be closed semialgebraic sets. Put $r = r(X) + r(Y)$ and $d = \max\{\kappa(X), \kappa(Y)\}$. Then there exists a positive constant C such that*

$$\operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geqslant C\left(\frac{\operatorname{dist}(x, X \cap Y)}{1 + |x|^d}\right)^{d(6d-3)^{n+r-1}} \quad \text{for } x \in \mathbb{R}^n.$$

Theorem 18 implies the following estimate of the global Łojasiewicz exponent in the Ji, Kollára and Shiffmana inequality [JKS] for semialgebraic mappings.

Theorem 19 ([38], Corollary 3.3) *Let $F : X \to \mathbb{R}^m$ be continuous semialgebraic mapping, where $X \subset \mathbb{R}^n$ is a closed set. If $d = \max\{\kappa(X), \kappa(Y)\}$ and $r = r(X) + r(Y)$, where $Y = \operatorname{graph} F$, then there exists a positive constant C such that*

$$|F(x)| \geqslant C\left(\frac{\operatorname{dist}(x, F^{-1}(0) \cap X)}{1 + |x|^d}\right)^{d(6d-3)^{n+m+r-1}} \quad \text{for } x \in X.$$

In particular, if the set X is unbounded and $F^{-1}(0) \cap X$ is compact, then

$$\mathcal{L}_\infty^{\mathbb{R}}(F|X) \geqslant (1-d)d(6d-3)^{n+m+r-1}.$$

In the paper [35] were used methods of K. Kurdyka, T. Mostowski and A. Parusiński (which were used in [KMP] for solution of the gradient conjecture of R. Thom). The results in the article [38] are a continuation of [35], by using methods of semialgebraic geometry.

3. Topologic types of mappings. To this topic are related articles [29], joint with T. Rodak, [40], joint with P. Migus and T. Rodak, review articles [45] and [48], joint with B. Osińska-Ulrych and G. Skalski, and partially [28], joint with T. Rodak. The last article will be discussed in the section about the trivialization of polynomials.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. René Thom in [Th] formulated a hypothesis, that in the set of polynomials $f \in \mathbb{K}[x_1, \ldots, x_n]$ in n variables $x_1, \ldots, x_n$, of degree $\deg f \leqslant k$, there exists a finite number of topological types, i. e. equivalence classes of the relation: $f \sim g$ iff $f \circ \varphi = \psi \circ g$ for some homeomorphisms $\psi : \mathbb{K} \to \mathbb{K}$ and $\varphi : \mathbb{K}^n \to \mathbb{K}^n$. The hypothesis was proved by T. Fukuda in [Fuk]. In the local case this problem corresponds to the C^0-sufficiency of jets. By a k-jet in the C^l class we mean a family of all C^l class functions in neighbourhoods of $0 \in \mathbb{R}^n$ called *C^l-realisations* of this jet, possesing the same Taylor polynomial of degree k at 0. The k-jet is said to be C^r-sufficient in the C^l class, if for every two of its C^l-realisations f and g there exists a C^r diffeomorphism φ of neighbourhoods of 0, such that $f \circ \varphi = g$ in a neighbourhood of 0 (R. Thom [Th]). N. H. Kuiper [Kui] and T. C. Kuo [Kuo$_1$, Kuo$_2$] proved the following criterion:

If the Łojasiewicz exponent at 0 of the gradient ∇f of C^k function f is less or equal $k-1$, then the k-jet of f is C^0-sufficient in the C^k class.

J. Bochnak and S. Łojasiewicz [BŁ] proved that, under the assumption $f(0) = 0$, the inverse to the above criterion is also true. We describe this in a more detailed way in review articles [45, 48]. Analogous results to the above, in the complex case, were proved by S. H. Chang and Y. C. Lu [CL], B. Teissier [Te] and J. Bochnak and W. Kucharz [BK].

This result was an inspiration for many authors to deal with the Łojasiewicz exponent and classification of singularities of functions, for example: F. Acquistapace, F. Broglia, M. Shiota [ABS], J. Bochnak, J. J. Risler, [BR], J. Chądzyński, T. Krasiński [CK$_4$, CK$_5$], E. Cygan, T. Krasiński, P. Tworzewski, [CyKT], J. Gwoździewicz [Gw], J. Kollár [Ko$_2$, Ko$_3$], M. Lejeune-Jalabert, B. Teissier [L-JT], B. Lichtin [Li], A. Melle-Hernndez [M-H], M. Merle [Mer], A. Płoski [Pł$_1$, Pł$_2$, Pł$_4$, Pł$_7$], B. Teissier [Te]).

a) ***Izotopicality of mappings at a point.*** The above Kuiper, Kuo and Bochnak-Łojasiewicza theorem concerns the isolated singularity of f at 0, i.e. the point 0 is an isolated zero of ∇f. The case of non-isolated singularity was investigated by many authors, for instance by J. Damon and T. Gaffney [DG], T. Fukui and E. Yoshinaga [FY], V. Grandjean [Gr], L. Kushner [Kush], Xu Xu [Xu] and for complex functions - by D. Siersma [Sie$_1$, Sie$_2$] and R. Pellikaan [Pe].

The purpose of this article is generalisation of the above results for a C^k mappings in a neighbourhood of zero with non-isolated singularity at zero.

The set of C^k class mappings $(\mathbb{R}^n, a) \to \mathbb{R}^m$ is denoted by $\mathcal{C}^k_a(n,m)$. By $j^k f(a)$ we denote the k-jet at a (in the class C^k) determined by a function $f \in \mathcal{C}^k_a(n,1)$. For a mapping $F = (f_1, \dots, f_m) \in \mathcal{C}^k_a(n,m)$ we put

$$j^k F(a) = (j^k f_1(a), \dots, j^k f_m(a)).$$

Let $Z \subset \mathbb{R}^n$ be a set such that $0 \in Z$ and let $k \in \mathbb{Z}$, $k > 0$. By *k-Z-jet in the class* $\mathcal{C}^k_0(n,m)$, or shortly *k-Z-jet*, we mean an equivalence class $w \subset \mathcal{C}^k_0(n,m)$ of the equivalence relation $\sim$: $F \sim G$ iff for some neighbourhood $U \subset \mathbb{R}^n$ of the origin, $j^k F(a) = j^k G(a)$ holds for $a \in Z \cap U$ (cf. [Xu]). The mappings $F \in w$ is called *C^k-Z-realisations of the jet w* and we write $w = j^k_Z F$. The set of all jets $j^k_Z F$ we denote by $J^k_Z(n,m)$.

The k-Z-jet $w \in J^k_Z(n,m)$ is said to be *C^r-Z-sufficient* (resp. *Z-v-sufficient*) in the class C^k, if for every of its C^k-Z-realisations f and g there exist sufficiently small neighbourhoods $U_1, U_2 \subset \mathbb{R}^n$ of 0, and a C^r diffeomorphism $\varphi : U_1 \to U_2$, such that $f \circ \varphi = g$ in U_1 (resp. there exists a homeomorphism $\varphi : [f^{-1}(0) \cup Z] \cap U_1 \to [g^{-1}(0) \cup Z] \cap U_2$) with $\varphi(0) = 0$ and $\varphi(Z \cap U_1) = Z \cap U_2$.

The following Kuiper and Kuo criterion for jets with non-isolated singularity was proved by Xu Xu [Xu].

Let $Z \subset \mathbb{R}^n$ be a closed set such that $0 \in Z$. If $f \in \mathcal{C}^k(n,1)$ such that $V(\nabla f) \subset Z$, satisfies the condition

$$(13) \qquad |\nabla f(x)| \geqslant C \operatorname{dist}(x, Z)^{k-1} \ \text{as } x \to 0 \text{ for some constant } C > 0,$$

then the k-Z-jet of f is C^0-Z-sufficient.

In the paper [40] we prove that the above theorem also holds for mappings. Let us start with some definitions and notations. Let X, Y be Banach spaces over $\mathbb{R}$. Let $L(X,Y)$ denote the Banach space of linear continuous mappings from X to Y. For $A \in L(X,Y)$, A^* stands for the adjoint operator to A in $L(Y', X')$, where X', Y' are the dual spaces of X and Y respectively. For $A \in L(X,Y)$ we investigate the Rabier function [Ra], namely,

$$(14) \qquad \nu(A) = \inf\{\|A^*\varphi\| : \varphi \in Y', \|\varphi\| = 1\},$$

where $\|A\|$ is the norm of a linear mapping A. In the case $f \in \mathcal{C}_0^k(n,1)$ we have $\nu(df) = |\nabla f|$, where df is the differential of the mapping f.

Theorem 20 ([40], Theorem 3) *Let $f\colon (\mathbb{R}^n,0) \to (\mathbb{R}^m,0)$, where $m \leqslant n$, be a C^k-Z-realisation of a k-Z-jet $w \in J_Z^k(n,m)$, where $k > 1$ and $Z = \{x \in \mathbb{R}^n : \nu(df(x)) = 0\}$, $0 \in Z$. Assume that for a positive constant C,*

$$\nu(df(x)) \geqslant C \operatorname{dist}(x,Z)^{k-1} \quad \text{as } x \to 0.$$

Then the jet w is C^0-Z-sufficient in the class C^k.

In Theorem [40, Theorem 3] we show more general assertion that for any C^k-Z-realisations f_1, f_2 of w, the deformation $f_1 + t(f_2 - f_1)$, $t \in \mathbb{R}$ is topologically trivial along $[0,1]$ (cf., [DG]). In particular the mappings f_1 and f_2 are isotopic at zero. We prove that Theorem 20 also holds for holomorphic mappings.

It is not clear for the authors whether there is the inverse theorem to the above. However we received the following theorem of Bochnak-Łojasiewicz type for functions (which is in certain sense an inverse theorem to the Xu Xu one).

Theorem 21 ([40], Theorem 4) *Let $Z \subset \mathbb{R}^n$, $n \geqslant 2$, be a set such that $0 \in Z$, let w be a k-Z-jet, $k > 1$, and let f be its C^k-Z-realisation. If w is Z-v-sufficient in C^k-class, $f(0) = 0$ and $V(\nabla f) \subset Z$, then the inequality* (13) *holds.*

In the case when Z is an algebraic or analytic set, some algebraic conditions for finite determinacy of a smooth function k-Z-jet were obtained by L. Kushner [Kush] and L. Kushner and B. Terra Leme [KLe]. In the above two papers, the authors use the idea of J. Mather [Mat] and J. C. Tougeron [To].

b) *Izotopicality of mappings at infinity.* Research of equivalence of complex polynomials in a neighborhood of infinity were conducted by P. Cassou-Noguès and H.V. Ha [CH] in the 2-dimensional case and by L. Fourrier [Fo] and G, Skalski [Sk] in the multi-dimensional case. In the paper [29] we generalize these results to the case of mappings of class C^2 defined in a neighbourhood of infinity. Instead of analytic equivalence being considered by the authors we consider isotopicality of mappings. Let's start with some notations and definitions.

Let $f\colon \Omega_1 \to \mathbb{R}^m$, $g\colon \Omega_2 \to \mathbb{R}^m$ where $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are neighbourhoods of infinity, i.e., their complements are compact sets in $\mathbb{R}^n$. We call f and g *isotopical at infinity* if there exist a neighbourhood of infinity $\Omega \subset \Omega_2$ and a continous mapping $H\colon \Omega \times [0,1] \to \Omega_1$ such that:

(a) $H_0(x) = x$, $x \in \Omega$,

(b) for any t the mapping H_t is a C^1 diffeomorphism and $\lim_{x\to\infty} H_t(x) = \infty$,

(c) $f(H_1(x)) = g(x)$, $x \in \Omega$,

where the mapping $H_t : \Omega \to \mathbb{R}^n$ is defined by $H_t(x) = H(x,t)$ for $x \in \Omega$, $t \in [0,1]$. The mapping H is also called *isotopy.*

By $\mathcal{P}_{k,\varepsilon}$, where $k \in \mathbb{R}$ and $\varepsilon > 0$, we denote the set of all C^2 mappings $P\colon \mathbb{K}^n \to \mathbb{K}^m$, for which there exists $R > 0$ such that

$$|P(x)| \leqslant \varepsilon |x|^k \text{ and } \|dP(x)\| \leqslant \varepsilon |x|^{k-1} \text{ for any } |x| \geqslant R,$$

where dP is the differential of P and $dP(x)$ the diferential of P at $x \in \mathbb{K}^n$.

The main result of thie paper [29] is the following

Theorem 22 ([29], Theorem 1) *Let $f\colon \mathbb{K}^n \to \mathbb{K}^m$, where $m \leqslant n$, be a C^2 mapping (holomorphic if $\mathbb{K} = \mathbb{C}$). Assume that there exist $k \in \mathbb{R}$ and positive constants C, R such that*

$$\nu(df(x)) \geqslant C|x|^{k-1}, \quad |x| \geqslant R,$$

where ν is the Rabier function. Then there exists $\varepsilon > 0$ such that for any $P \in \mathcal{P}_{k,\varepsilon}$ the mappings f and $f + P$ are isotopical at infinity.

G. Skalski [Sk] proved that the inverse of the above theorem is false, even for functions. Therefore, the situation at infinity is different from the case of jets.

It is worth noting that in the case of complex polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^n$, the exponent $\mathcal{L}_\infty(\nu(df))$ is finite if and only if the jacobian of f is a nonzero constant.

c) ***Trivializtion of a polynomial near the fibre of a mapping.*** The paper [28] and the review one [47] are devoted to conditions for trivialization of a polynomial in a neighbourhood of the polynomial fibre.

We say that a point $\lambda \in \mathbb{C}$ is a *typical value* of a polynomial $f : \mathbb{C}^n \to \mathbb{C}$ if there exists a neighbourhood $U \subset \mathbb{C}$ of λ such that the function $f : f^{-1}(U) \to U$ is a trivial C^∞ fibration. A point λ which is not typical is called a *bifurcation point* of f. The set of bifurcation points of f we denoty by $B(f)$. J. L. Verdier [Ve] proved that the set of bifurcation points of a regular mapping is contained in some proper algebraic subset of counterdomain. For a polynomial it means that the set $B(f)$ is finite. Obviously we have $B_\infty(f) \subset B(f)$, where $B_\infty(f)$ is the set of bifurcation points of f at infinity (defined previously) It is known that $B(f) = B_\infty(f) \cup C(f)$, where $C(f)$ is the set of critical values of f. Analogously as above we define the set of bifurcation points of a real function $f : \mathbb{R}^n \to \mathbb{R}$.

The problem of trivialization of polynomials is of interest for many mathematicians, among others: J. Chądzyński [Ch], J. Chądzyński and T. Krasiński [CK$_4$], T. Krasiński [Kra$_1$], J. Gwoździewicz and A. Płoski [GwP], H. V. Ha [Ha], Z. Jelonka [J$_3$], Z. Jelonek and K. Kurdyka [JK2], Z. Jelonek and M. Tibăr [JT], K. Kurdyka, P. Orro and S. Simon [KOS], A. Némethi and A. Zaharia [NZ], A. Parusiński [Pa$_2$], A. N. Varchenko [Va].

One of the metric conditions that lead to trivialization at infinity of a polynomial in a neighbourhood of the fibre $f^{-1}(\lambda)$ is the following *Malgrange condition*:

$$\text{(M)} \qquad |\nabla f(z)| \geqslant \delta |z|^{-1} \qquad \text{for} \qquad |z| \geqslant R, \quad |f(z) - \lambda| \leqslant \varepsilon,$$

where $R,\ \varepsilon,\ \delta > 0$. This condition implies the trivialization of a polynomial (cf. [Pa$_2$]), namely:

Let $\lambda \in \mathbb{C}$. If (M) *holds, then there is a trivialisation at infinity of the polynomial f over $U = \{\xi \in \mathbb{C} : |\xi - \lambda| < \varepsilon\}$.*

Păunescu and Zaharia [PZ] showed that this theorem can not be inverted. In the review article [47] we discuss in detail the relationship between the Malgrange condition in a neighbourhood of the fibre and a trivialisation of a polynomial over neighbourhood of a point.

The above theorem and the condition (M) lead to the notion of generalized critical values of a polynomial and lead to the next version of the Łojasiewicz exponent, studied in [28].

Let M, N, L be finite-dimensional real vector spaces, $X \subset M$ be a closed semi-algebraic set, $g : X \to N$ and $f : X \to L$ be continuous semi-algebraic mappings (see [BR, BCRo]), and let $\lambda \in L$. By the *Łojasiewicz exponent at infinity of g near the fibre $f^{-1}(\lambda)$*, we call the upper bound of the exponents θ for which the following *Łojasiewicz inequality* holds

$$(\text{Ł}) \qquad |g(x)| \geqslant C|x|^\theta \quad \text{as} \quad x \in X, \quad |x| \to \infty \quad \text{and} \quad f(x) \to \lambda$$

with $C > 0$ (cf. [Ł_3, Ł_4], [Ro_2]). We denote this exponent by $\mathcal{L}_{\infty,f\to\lambda}(g)$.

The main result of the article [28] is the following theorem.

Theorem 23 ([28], Theorem 1.2) *Let $g : X \to N$ and $f : X \to L$ be continuous semi-algebraic mappings.*

(i) *For any $\lambda \in L$, $\mathcal{L}_{\infty,f\to\lambda}(g) \in \mathbb{Q} \cup \{-\infty, +\infty\}$.*

(ii) *The function*

$$\vartheta_{g/f} : L \ni \lambda \mapsto \mathcal{L}_{\infty,f\to\lambda}(g)$$

is upper semi-continuous, and there exists a semi-algebraic stratification $L = S_1 \cup \cdots \cup S_j$ such that $\vartheta_{g/f}$ is constant on each stratum S_i, $i = 1, \ldots, j$.

The key points in the proof are Lipschitz stratifications introduced by T. Mostowski [Mos] (see also [Pa_1]) and properties of the set of points at which a mapping is not proper, considered by Z. Jelonek [J_1].

If $f : M \to L$ is a semi-algebraic mapping of class C^1, we define the *Łojasiewicz exponent of df near the fibre $f^{-1}(\lambda)$* by

$$\mathcal{L}_{\infty,\lambda}(f) = \mathcal{L}_{\infty,f\to\lambda}(\nu(df)),$$

where ν is the Rabier function (14). This notion was introduced by H. V. Ha [Ha] in the case of complex polynomial functions in two variables. The Rabier function $\nu(df)$ for polynomials coincides with the norm of gradient $|\nabla f|$ of f. In this case, the notion was carefully researched by J. Chądzyński and T. Krasiński (see for instance [CK_8]) and by J. Gwoździewicz and A. Płoski (see for instance [GwP]).

The exponent $\mathcal{L}_{\infty,\lambda}(f)$ is strongly related to the set of bifurcation points of f. Namely, one can define the *set of generalized critical values* of f by

$$K_\infty(f) = \{\lambda \in L : \mathcal{L}_{\infty,\lambda}(f) < -1\}.$$

It is a closed and semi-algebraic set. By Theorem 23, the mapping $L \ni \lambda \mapsto \mathcal{L}_{\infty,\lambda}(f)$ has a finite number of values, hence there exists $\alpha > 0$ such that

$$K_\infty(f) = \{\lambda \in L : \mathcal{L}_{\infty,\lambda}(f) < -1 - \alpha\}.$$

It is known that if $f : M \to \mathbb{R}$ is of class C^2, then $B_\infty(f) \subset K_\infty(f)$, where $B_\infty(f)$ denotes the set of bifurcation points at infinity of f (the definition of this set is analogous to the one for complex polynomials). In the case of polynomials, $K_\infty(f)$ is always a finite set (see for instance [Ph_2, Ra, KOS, Wa, Va, Ve, Ha, Pa_2]). Estimates of the number of points of this set in terms of the degree of a polynomial gave Z. Jelonek and K. Kurdyka [JK1, JK2]. For complex polynomials with a finite number of singular points at infinity there is $B_\infty(f) = K_\infty(f)$ (see [Ha] in the two-dimensional case, and [Pa_2] in the general case).

Chądzyński and Krasiński [CK_8] proved that for a complex polynomial f in two variables with $\deg f > 0$ there exists $c_f \in \mathbb{Q}$ with $c_f \geqslant 0$ such that

$$\mathcal{L}_{\infty,\lambda}(f) = c_f \text{ for } \lambda \notin K_\infty(f) \quad \text{and} \quad \mathcal{L}_{\infty,\lambda}(f) < -1 \text{ for } \lambda \in K_\infty(f).$$

They also asked whether $\lambda \mapsto \mathcal{L}_{\infty,\lambda}(f)$ behaves similarly in the general case. Note that in the multi-dimensional case we cannot require $c_f \geqslant 0$. Indeed, for the polynomial $f(z_1, z_2, z_3) = (z_1z_2 - 1)z_2z_3$ ([Ra]) we have $c_f = -1$ (see [CK_8]).

As a corollary from Theorem 23 we get a partial answer to the above mentioned question. Namely, for any polynomial $f : \mathbb{C}^n \to \mathbb{C}$, with positive degree, there exist a finite set $S \subset \mathbb{C}$ with $K_\infty(f) \subset S$ and $c_f \geqslant -1$ such that $\mathcal{L}_{\infty,\lambda}(f) = c_f$ for $\lambda \in \mathbb{C} \setminus S$ and $\mathcal{L}_{\infty,\lambda}(f) < -1$ for $\lambda \in S$ ([28, Corollary 1.7]).

4. Sums of squares and optimization. In the paper [37], joint with K. Kurdyka, we study two types of problems for polynomials which are positive (or nonnegative) on subsets of $\mathbb{R}^n$. In the first part we prove stronger and effective versions of the known approximation and representation theorems with sums of squares of polynomials. Next we give quantitative versions of these results and we explain some applications to semidefinite optimization methods. In the second part we prove a theorem on convexification of a polynomial, i.d. any polynomial f which is positive on a convex closed set X becomes strongly convex when multiplied by $(1 + |x|^2)^N$ with N large enough (the noncompact case requires some extra assumptions). In fact we give an explicit estimate for N, which depends on the size of the coefficients of f and on the lower bound of f on X. As an application of our convexification method we propose an algorithm which for a given polynomial f on a compact convex semialgebraic set X produces a sequence (starting from an arbitrary point in X) which converges to a lower critical point of f on X.

a) *Sum of squares and approximation.* We denote by $\mathbb{R}[x]$ the ring $\mathbb{R}[x_1, \dots, x_n]$ of polynomials in $x = (x_1, \dots, x_n)$ with coefficients in $\mathbb{R}$. Important problems of real algebraic geometry are representations of nonnegative polynomials on closed semialgebraic sets. Recall Hilbert's 17th problem (solved by E. Artin [Ar]): *if $f \in \mathbb{R}[x]$ is nonnegative on $\mathbb{R}^n$, then*

$$\text{(AH)} \qquad fh^2 = h_1^2 + \cdots + h_m^2 \quad \text{for some } h, h_1, \dots, h_m \in \mathbb{R}[x],\ h \neq 0,$$

that is, f is a sum of squares of rational functions. D. Hilbert [Hil] proved that for $n \geqslant 2$ there exist nonnegative polynomials which are not sums of squares of polynomials. Not until 1967 T. S. Motzkin gave an explicit example of such a polynomial, $f(x_1, x_2) = 1 + x_1^2x_2^2(x_1^2 + x_2^2 - 3)$. In some cases are known forms of functions h and h_j in (AH). For instance B. Reznick [Re_1, Theorem 3.12] proved that:

With the additional assumptions that f is homogeneous and $f(x) > 0$ for $x \neq 0$, there exists an nonnegative integer r_0 such that for any $N \geqslant r_0$ the polynomial $(x_1^2 + \cdots + x_n^2)^N f(x)$ is a sum of even powers of linear functions. Reznick gave also effective estimate of the number r_0.

Let $X \subset \mathbb{R}^n$ be a *closed basic semialgebraic set* defined by $g_1, \dots, g_r \in \mathbb{R}[x]$, i.e.,

$$X = \{x \in \mathbb{R}^n : g_1(x) \geqslant 0, \dots, g_r(x) \geqslant 0\}. \tag{15}$$

The *preordering* generated by $g_1, \dots, g_r$ is defined to be

$$T(g_1, \dots, g_r) = \Big\{ \sum_{e=(e_1,\dots,e_r)\in\{0,1\}^r} \sigma_e g_1^{e_1} \cdots g_r^{e_r} : \sigma_e \in \sum \mathbb{R}[x]^2 \text{ for } e \in \{0,1\}^r \Big\},$$

where $\sum \mathbb{R}[x]^2$ denotes the set of sums of squares of polynomials from $\mathbb{R}[x]$. Natural generalizations of the above theorem of Artin are the Stellensätze by J.-L. Krivine [Kri], D. W. Dubois [Du$_2$], and J.-J. Risler [Ri] (see also [BCRo, Sche$_5$, Mar$_2$, PD]). When the set X is compact, a very important result was obtained by K. Schmüdgen (see [Schm$_1$], [CMN]):

Every strictly positive polynomial f on X belongs to the preordering $T(g_1, \dots, g_r)$.

Under the assumptions that the preordering $T(g_1, \dots, g_r)$ is archimedean, M. Putinar [Pu] proved that:

Every strictly positive polynomial f on X belongs to the quadratic module *generated by $g_1, \dots, g_r$,*

$$P(g_1, \dots, g_r) := \Big\{ \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_r g_r : \sigma_i \in \sum \mathbb{R}[x]^2,\ i = 0, \dots, r \Big\}.$$

For over ten years intensive studies are conducted on the making use of these facts to minimize polynomials on semialgebraic sets. One of the barriers in this regard is the difficulty in effective estimating the degrees of polynomials in the Schmüdgen representation

$$f = \sum_{e\in\{0,1\}^r} \sigma_e g_1^{e_1} \cdots g_r^{e_r} \in T(g_1, \dots, g_r). \tag{16}$$

M. Schweighofer [Schw$_2$] obtained an upper bound estimate for $\deg \sigma_e g_1^{e_1} \cdots g_r^{e_r}$ in terms of $\deg f$, $f^* := \min\{f(x) : x \in X\}$ and the coefficients of f, provided that $f^* > 0$. Estimation of these degrees, passing over the f^*, would reduce the problem of minimizing of a polynomial to finite-dimensional space of polynomials. This is an issue so far unsolved, and in some cases it is known that it does not exists. As shown by C. Scheiderer [Sche$_4$], *there is no such bound in terms of* $\deg f$ *unless* $\dim(X) \leqslant 1$.

The above Schmüdgen and Putinar results concern strictly positive polynomials (on X). In the case of nonnegative polynomials C. Berg, J. P. R. Christensen and P. Ressel [BCRo] and J. B. Lasserre and T. Netzer [LN, Corollary 3.3] proved that:

Any polynomial f which is nonnegative on $[-1,1]^n$ can be approximated in the l_1-norm by sums of squares of polynomials.

The l_1-norm of a polynomial is defined to be the sum of the absolute values of its coefficients (in the usual monomial basis). In this connection J. B. Lasserre [La$_4$, Theorem 2.6] (see also [La$_3$]) proved that:

If $g_1, \dots, g_r$ are concave polynomials such that $g_1(z) > 0, \dots g_r(z) > 0$ for some $z \in X$, then any convex on $\mathbb{R}^n$ polynomial nonnegative on X can be approximated in the l_1-norm by polynomials from the cone

$$L_c(g_1, \dots, g_r) := \Big\{ \sigma_0 + \lambda_1^2 g_1 + \cdots + \lambda_r^2 g_r : \sigma_0 \in \sum \mathbb{R}[x]^2 \text{ convex},\ \lambda_1, \dots, \lambda_r \in \mathbb{R} \Big\}.$$

In the paper [37] we prove a theorem similar to the Schmüdgen and Putinar results for the following cone:

$$\mathcal{S}(g_1,\dots,g_r)=\{\sigma_0+\varphi(g_1)g_1+\cdots+\varphi(g_r)g_r:\sigma_0\in\sum\mathbb{R}[x]^2,\ \varphi\in\sum\mathbb{R}[t]^2\},$$

where t is a single variable. Obviously $\mathcal{S}(g_1,\dots,g_r)\subset P(g_1,\dots,g_r)$. Namely, we have

Theorem 24 ([37], Theorem 2.1) *Let $f\in\mathbb{R}[x]$ be nonnegative on the set X. Then there exists a sequence $f_\nu\in P(g_1,\dots,g_r)$, $\nu\in\mathbb{N}$, that is uniformly convergent to f on compact subsets. Moreover, f_ν can be chosen from the cone $\mathcal{S}(g_1,\dots,g_r)$. In particular f_ν converges to f in the l_1-norm.*

We show also that for approximation of a function f by polynomials from the cone $\mathcal{S}(g_1,\dots,g_r)$, it enough to take the functions φ in the form $\varphi(t)=(at+b)^{2N}$, where $a,b\in\mathbb{R}$.

Consider the next positive cone:

$$\mathcal{K}(g,g_1,\dots,g_r):=\Big\{\sigma_0+\sigma_1 g+\varphi(g_1)g_1+\cdots+\varphi(g_r)g_r:\sigma_0,\sigma_1\in\sum\mathbb{R}[x]^2,\\ \varphi\in\sum\mathbb{R}[t]^2\Big\},$$

where $g\in\mathbb{R}[x]$. Note that if we put

$$\Phi(g_1,\dots,g_r):=\left\{\varphi(g_1)g_1+\cdots+\varphi(g_r)g_r:\varphi\in\sum\mathbb{R}[t]^2\right\},$$

then

$$\mathcal{K}(g,g_1,\dots,g_r)=T(g)+\Phi(g_1,\dots,g_r),$$

From Schmüdgen Theorem and Theorem 24 we obtain the following wersion of the Schmüdgen and Putinar results.

Theorem 25 ([37], Corollary 3.1) *Assume that X is a compact set. Let $R>0$ be a number such that the polynomial $g_0(x)=R^2-|x|^2$ is nonnegative on X. If $f\in\mathbb{R}[x]$ is strictly positive on the set X, then $f\in\mathcal{K}(g_0,\dots,g_r)$.*

Just as in Theorem 24, to represent the function f as an element of $\mathcal{K}(g_0,\dots,g_r)$, it is enough just to consider polynomials $\varphi(t)=(at+b)^{2N}$, where $a,b\in\mathbb{R}$. By the above, Theorem 25 shows what form are the sums of squares appearing in the Schmüdgen (16) representation and simplifies the procedure of calculation of this representation.

We initiated in the paper [34] joint with K. Kurdyka, B. Osińska-Ulrych and G. Skalski tudies on the Positivstellensatz for a positive cone of an unbounded algebraic set. In this article, we gave a constructive proof of C. Scheiderer theorem [Sche$_2$] about extending of a polynomial positive on an unbounded algebraic set $V\subset\mathbb{R}^n$, $n\geqslant 2$, to a polynomial positive on the ambient space $\mathbb{R}^n$. More precisely:

If a set $V\subset\mathbb{R}^n$ defined by the system of polynomial equations $h_1(x)=\cdots=h_r(x)=0$ and a polynomial $f:\mathbb{R}^n\to\mathbb{R}$ is positive on V, then there exists a polynomial of the form $h(x)=\sum_{i=1}^r h_i^2(x)\sigma_i(x)$, where $\sigma_i\in\sum\mathbb{R}[x]^2$, such that $f(x)+h(x)>0$ for $x\in\mathbb{R}^n$.

We also give the form of polynomials σ_i and effective estimation of their degrees in terms of degrees of f, h_i and the Łojasiewicz exponent $\mathcal{L}_\infty(f|V)$ ([34, Theorem 4.1]).

b) ***Semidefinite optimization.*** Let X be a compact set of the form (15). In [La$_1$] Lasserre gave a method of minimizing a polynomial f on the set X in terms of the cone $P(g_1, \ldots, g_r)$. More precisely, let

$$f^* := \inf\{f(x) : x \in X\}.$$

Then $f^* = \sup\{a \in \mathbb{R} : f(x) - a > 0 \text{ for } x \in X\}$, and by Putinar's result [Pu],

$$f^* = \sup\{a \in \mathbb{R} : f - a \in P(g_1, \ldots, g_r)\},$$

and $f^* = \inf\{L(f) : L : \mathbb{R}[x] \to \mathbb{R} \text{ is linear}, L(1) = 1, L(P(g_1, \ldots, g_r)) \subset [0, \infty)\}$. Since the cone $P(g_1, \ldots, g_r)$ is contained in an infinite dimensional vector space $\mathbb{R}[x]$ over $\mathbb{R}$, so it is difficult to solve such a possed problem. Lasserre proposed a method of reducing the dimension (Lasserre relaxation) by applying the following cones

$$P_k(g_1, \ldots, g_r) := \Big\{\sigma_0 g_0 + \cdots + \sigma_r g_r \in P(g_1, \ldots, g_r) : \deg \sigma_i g_i \leqslant k,\ i = 0, \ldots, r\Big\},$$

where we put $g_0 = 1$, and by reducing the problem to calculating the sequences

$$\begin{aligned} a_k^* &:= \sup\{a \in \mathbb{R} : f - a \in P_k(g_1, \ldots, g_r)\}, \\ l_k^* &:= \inf\{L(f) : L : \mathbb{R}[x]_k \to \mathbb{R} \text{ is linear}, L(1) = 1, L(P_k(g_1, \ldots, g_r)) \subset [0, \infty)\}, \end{aligned}$$

for sufficiently large $k \in \mathbb{N}$, where $\mathbb{R}[x]_k$ is the linear space of polynomials $h \in \mathbb{R}[x]$ such that $\deg h \leqslant k$. Lasserre proved that

(a_k^*), (l_k^*) *are increasing sequences that converge to* f^* *and* $a_k^* \leqslant l_k^* \leqslant f^*$ *for* $k \in \mathbb{N}$.

Theorem 25 allows us to apply the Lasserre algorithm by using the cones

$$\begin{aligned} \mathcal{K}_k(g, g_1, \ldots, g_r) := \{\sigma_0 + \sigma_1 g + \varphi(g_1) g_1 + \cdots + \varphi(g_r) g_r \in \mathcal{K}(g, g_1, \ldots, g_r) : \\ \deg \sigma_0, \deg \sigma_1 g, \deg g_i \varphi(g_i) \leqslant k\}, \quad k \in \mathbb{N}, \end{aligned}$$

instead of $P_k(g, g_1, \ldots, g_r)$. Namely, we have

Theorem 26 ([37], Remark 3.2) *Denote*

$$\begin{aligned} u_k^* &:= \sup\{a \in \mathbb{R} : f - a \in \mathcal{K}_k(g, g_1, \ldots, g_r)\}, \\ v_k^* &:= \inf\{L(f) : L : \mathbb{R}[x]_k \to \mathbb{R} \textit{ is linear}, L(1) = 1, L(\mathcal{K}_k(g, g_1, \ldots, g_r)) \subset [0, \infty)\}, \end{aligned}$$

for sufficiently large $k \in \mathbb{N}$. *We see that* (u_k^*), (v_k^*) *are increasing sequences that converge to* f^* *and* $u_k^* \leqslant v_k^* \leqslant f^*$ *for* $k \in \mathbb{N}$.

Consideration of the cones $\mathcal{K}_k(g, g_1, \ldots, g_r)$ potentially simplifies the problem of minimizing polynomials on the set X, since these cones are properly contained in $P_k(g, g_1, \ldots, g_r)$, and consequently $a_k^* \leqslant u_k^*$ and $l_k^* \leqslant v_k^*$ for sufficiently large $k \in \mathbb{N}$.

c) ***Approximation of polynomials on compact semialgebraic sets.*** In the paper [37], we present another method of approximate minimizing a polynomial f on a compact basic semialgebraic set X, say $X \subset \{x \in \mathbb{R}^n : |x| \leqslant R\}$, by reducing the problem to the case $X = \{x \in \mathbb{R}^n : |x| \leqslant R\}$. Namely, in [37, Proposition 3.3] we prove that:

For any $\epsilon > 0$, we give an effective procedure for calculating a polynomial $h \in \Phi(g_1, \dots, g_r)$ such that

$$f^* - 2\epsilon \leqslant \inf\{f(y) - h(y) : |y| \leqslant R\} \leqslant f^* + 2\epsilon,$$

where $f^* := \inf\{f(x) : x \in X\}$, while calculate the polynomial h is enought to take a polynomial φ of the form as in Theorem 25.

Thus, the problem of approximate minimization of f can be reduced to the simpler case when the set X is described by one inequality $R^2 - |x|^2 \geqslant 0$. In this case M. Schweighofer [Schw$_2$] gave the rate of convergence of the sequence

$$a_k^{**} := \sup\{a \in \mathbb{R} : f - h - a \in P_k(R^2 - |y|^2)\} \to f^{**}, \quad \text{as } k \to \infty,$$

where $f^{**} := \inf\{f(y) - h(y) : |y| \leqslant R\}$, by

$$f^{**} - a_k^{**} \leqslant \frac{c}{\sqrt[D]{k}}$$

for some constant $c \in \mathbb{N}$ dependent on f and $R^2 - |y|^2$ and the constant $D \in \mathbb{N}$ dependent of $R^2 - |y|^2$. These constants depend on the exponent and the constant C in the Łojasiewicz inequality (Ł$_2$). This estimate is important from the point of view of computer implementation of this algorithm.

In the above discussed approximation, for the construction of the polynomial h, the basic role is played by an estimate of the distance of a point x from the set of zeros X of the semialgebraic function $G(x) = \max\{0, -g_1(x), \dots, -g_r(x)\}$, $x \in \mathbb{R}^n$, under given value $G(x)$. This problem can be solved by using an estimate of the Łojasiewicz exponent $\mathcal{L}$ and an estimate of the constant $C > 0$ in the following inequality

$$G(x) \geqslant C \left(\frac{\operatorname{dist}(x, X)}{1 + |x|^d} \right)^{\mathcal{L}}, \quad x \in \mathbb{R}^n.$$

Namely in [38, Corollary 10] we showed that

$$\mathcal{L} \leqslant d(6d - 3)^{n+r-1},$$

where $d = \max\{\deg g_1, \dots, \deg g_r\}$, and r is the number of inequalities needed to determine the set X. In view of the Bröcker estimate (8), we can always assume that $r \leqslant n(n+1)/2$. Estimates of the Łojasiewicz exponent and constant in the Łojasiewicz inequality also play a fundamental role in a similar study carried out by M. Schweighofer [Schw$_2$, Schw$_3$].

d) *Convexifying of polynomials.* Let $f \in \mathbb{R}[x]$, and let

$$\varphi_N(x) = (1 + x_1^2 + \dots + x_n^2)^N f(x), \quad N \in \mathbb{N}.$$

One of the main results of the paper [37] is the following theorem on convexifying positive polynomials on convex and compact sets.

Theorem 27 ([37], Theorem 5.1) *Assume that a polynomial f is strictly positive on a compact and convex set $X \subset \mathbb{R}^n$ containing at least two points. Put $R = \max\{|x| : x \in X\}$ and let*

$$0 < m \leqslant \min\{f(x) : x \in X\}. \tag{17}$$

Then there exists a uniquely determined natural number $\mathcal{N}$, dependent on the modules of coefficients of polynomials $f, g_1, \dots, g_r$ and the numbers R and m, such that for any natural number $N \geqslant \mathcal{N}$, the polynomial φ_N is strongly convex on X.

We also prove a theorem similar to Theorem 27 for unbounded sets (see [37, Theorem 6.5]). We also show that the Reznick theorem [Re_1, Theorem 3.12] mentioned earlier, can be in a sense inverted (see [37, Corollary 6.8]).

Theorem 27 implies that there exists a natural number N such that all functions of the form $\varphi_{N,\xi}(x) := (1+|x-\xi|^2)^N f(x)$, $\xi \in X$, are strongly convex on X. Moreover, the problem can be reduced to the case when $N = 6$ (see [37, formula (4.2)]).

The above observation allows to use Theorem 27 to indicate critical points of functions, which is an important issue in the optimization, concerning the calculating critical points of functions. It allows us to give an algorithm for determining a sequence of points of a convex and compact semialgebraic set X whose limit is a lower critical point of the polynomial f on a set X. Recall two definitions and some notations.

It is known that any strictly convex, hence in particular any strongly convex, function φ on a convex compact set X admits a unique point, denoted by $\operatorname{argmin}_X \varphi$, at which φ attains its minimum on X.

Choose an arbitrary point $a_0 \in X$, and by induction put

$$a_\nu := \operatorname{argmin}_X \varphi_{N,a_{\nu-1}}. \tag{18}$$

Let f be a C^1 function in a neighborhood U of a closed set $X \subset \mathbb{R}^n$. Recall that $a \in X$ is a *lower critical point of f on X* if

$$\langle \nabla f(a), x-a\rangle \geqslant 0 \quad \text{for } x \in X \ \text{ in a neighbourhood of } a.$$

Theorem 28 ([37], Theorem 7.5) *If $X \subset \mathbb{R}^n$ is a compact convex semialgebraic set, and f is a polynomial which is strictly positive on X, then the sequence defined by* (18) *is convergent and its limit is a lower critical point of f on X.*

In the above theorem, the assumption of strict positivity of the polynomial f on the set X can always be obtained by adding to the polynomial f appropriate constant, which can be effectively identified. It is also worth noting that we do not lose convergence of a_ν to a lower critical point of the function f, if we set that sequence in an approximate way.

e) *Homogeneous polynomials and sums of squares.* The article [36], joint with A. Gala-Jaskórzyńska, K. Kurdyka and K. Kuta, refers to similar issues in the previous section. In this paper we give an explicit form of representation of a positive polynomial on an unbounded semialgebraic set in M. Putinar and F. Vasilescu Positivstellensatz (see [PV_1, PV_2], cf., [Pu]). Recall this Positivstellensatz, under notations consistent with the others in the scientific report.

Let $(g_1, \dots, g_r)$ be an r-tuple of polynomials from the ring $\mathbb{R}[x]$, where $x = (x_1, \dots, x_n)$. Let $f \in \mathbb{R}[x]$. Suppose that the degrees of g_j's and f are all even. Let $G_1, \dots, G_r$, $F \in \mathbb{R}[x_0, \dots, x_n]$ be the homohenizations of the polynomials $g_1, \dots, g_r, f$ respectively and assume that

$$F(y) > 0 \quad \text{for } y \in \{y \in \mathbb{R}^{n+1} : G_i(y) \geqslant 0,\ i = 1, \dots, r\},\ y \neq 0. \tag{19}$$

Then there exist an integer $b \geqslant 0$ and a finite collection of real polynomials $q_l, q_{kl}, l \in L, k = 1, \dots, m,$ such that

$$f(x) = (1+|x|^2)^{-2b}\left(\sum_{l\in L} q_l(x)^2 + \sum_{k=1}^{m}\sum_{l\in L} g_k(x) q_{kl}(x)^2\right), \quad x \in \mathbb{R}^n. \tag{20}$$

This theorem is a strengthening of the Schmüdgens [Schm$_1$] and Putinar [Pu] Positivstellensatz in the case when the set X defined by the formula (15) is not compact. An assumption of strict positivity of the polynomial f on the set X is insufficient to provide the assumption (19). This assumption is equivalent to that $f(x) > 0$ for $x \in X$ and that the leading form $\widehat{f}(x) := F(0, x_1, \dots, x_n)$ of f takes positive values on the part of the set X lying on the hyperplane at infinity, ie. on the set $\widehat{X} = \{x \in \mathbb{R}^n \setminus \{0\} : \widehat{g}_i(x) \geqslant 0,\ i = 1, \dots, r\}$.

The aim of the article [36] is to simplify the assertion (20), and to give an explicit form of representation (20). For this purpose, at first we prove Theorem 29 for homogeneous polynomials. By $Q_{n,k}$ we denote the set of finite sums of k th powers of linear functions.

Theorem 29 ([36], Theorem 3) *Let $f \in \mathbb{R}[x]$ be a homogeneous polynomial of positive even degree d, and let $g_1, \dots, g_r \in \mathbb{R}[x]$ be homogeneous polynomials of even degrees and let X be a set defined by* (15). *If $f(x) > 0$ for $x \in X \setminus \{0\}$, then there exist positive even integers D, N, a polynomial $q \in Q_{n,D}$ and $a, b \in \mathbb{R}$ such that*

$$f(x) = |x|^{-D+d} \left(q + \sum_{i=1}^{r} |x|^{\alpha_i} (a g_i(x) + b|x|^{\deg g_i})^N g_i(x) \right), \tag{21}$$

where $\alpha_i = D - (N+1) \deg g_i$ for $i = 1, \dots, r$ are nonnegative even numbers.

In the proof of the above theorem we use a polynomial approximation method elaborated in [37] and the B. Reznick theorem [Re$_1$, Theorem 3.12].

An explicit form of Putinar-Vasilescu representation is the following theorem.

Theorem 30 ([36], Corollary 1) *Under the assumptions and notations of Putinar and Vasilescu Positivstellensatz there are even integers $b, D, N \geqslant 0$ such that $D-(N+1)\deg g_k \geqslant 0$ for $k = 1, \dots, r$, and a finite family of real polynomials q_l, $l \in L$, with $\deg q_l \leqslant 1$ and polynomials $q_{k,1}$, $k = 1, \dots, r$, of the form*

$$q_{k,1}(t) = (1 + |t|^2)^{\alpha_k} \left(\xi g_k(t) + \eta (1 + |t|^2)^{\frac{\deg g_k}{2}} \right)^N$$

for some $\xi, \eta \in \mathbb{R}$, where $\alpha_k = \frac{D-(N+1)\deg g_k}{2}$ for $k = 1, \dots, r$, such that

$$f(x) = (1 + |x|^2)^{-b} \left(\sum_{l \in L} q_l^D(x) + \sum_{k=1}^{r} g_k(x) q_{k,1} \right), \quad t \in \mathbb{R}^n.$$

Theorems 29 and 30 provide an additional information about how the polynomials defining the semialgebraic set X are involved in the representation of f.

f) ***Stability of algebras of bounded polynomials.*** This subject is dealt within the article [33], joint with M. Michalska and K. Kurdyka. In this paper we present a relation between bifurcation values at infinity of a polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ and algebras of polynomials bounded on $S_c = \{x \in \mathbb{R}^2 \mid f(x) \leqslant c\}$, $c \in \mathbb{R}$, $c > 0$.

Denote by $\mathcal{A}(S_c)$ the set of polynomials from $\mathbb{R}[x, y]$, which are bounded on the set S_c. The main result of this paper is the following theorem.

Theorem 31 ([33], Theorem 3.1) *Take any polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ and numbers $c, \tilde{c} \in \mathbb{R}$ such that $0 < c < \tilde{c}$. If $[c, \tilde{c}] \cap B_\infty(f) = \emptyset$, then $\mathcal{A}(S_c) = \mathcal{A}(S_{\tilde{c}})$.*

We may consider a polynomial $f-a$, where $a \in \mathbb{R}$, instead of f, and then the above result also holds in the case of algebras of polynomials bounded on sets $\{x \in \mathbb{R}^2 : a \leqslant f(x) \leqslant b\}$, where $a \leqslant b$ as well as for sets of the form $\mathcal{A}(\{x \in \mathbb{R}^2 : f(x) \leqslant b\})$.

In the above theorem we consider real polynomial f, but the set $B_\infty(f)$ is the set of bifurcation points of this polynomial treated as a complex polynomial. It does not change the fact that it is a finite set. As it was schown by Z. Jelonek and K. Kurdyka [JK1], the set has at most $d^{n-1}-1$ points, where d is the degree of f and n is the number of variables (in our case $n=2$). Therefore, there exist at most $2d-1$ different algebras $\mathcal{A}(\{x \in \mathbb{R}^2 : f(x) \leqslant b\}$ for $b \in \mathbb{R}$, where $d = \deg f$.

The crucial role in the proof of Theorem 31 was played by Puiseux theorem at infinity with a parameter due to T. Krasiński [Kra_1, Kra_2] and by comparising norms in the space of real polynomials in one variable.

Researches on rings of bounded polynomials on semialgebraic sets are conducted recently. They were studied for example by E. Becker and V. Powers [BP], C. Scheiderer [Sche_3] M. Schweighofer [Schw_1] K. Schmüdgen [Schm_2], S. Kuhlmann and M. Marshall [KM]. An important inspiration for this research was the result of K. Schmüdgena [Schm_2], which we described earlier. It concern the moment problem on compact semialgebraic sets (see also results by S. Kuhlmann and M. Marshall [KM]) and shows the relationship between the algebra of bounded polynomials and optimization (see, for example [Mar_2]).

5. Real fields. In the paper [32] we give an elementary geometric construction of any real closed field in terms of field of Nash functions. We also give a characterization of any Archimedean field in terms of fields of Nash functions.

In the study of the 17th Hilbert problem the orderings of a real field k are of importance (see [Al], [AGR], [Ar], [AS], [BE], [Bröc_1], [Du_2], [Gu], [Mar_1], [PD]). By the Artin-Schreier theorem [AS], the study of such orderings amounts to considering real closures of k. In the paper [32] we give a geometric construction of a universal model of an arbitrary real closed field. To this end we construct, in terms of Nash functions, all real closures of the rational functions field $k = \mathbb{Q}(\Lambda_T)$, where $\Lambda_T = (\Lambda_t : t \in T)$, $T \neq \emptyset$, is a system of any number of variables. This suffices to achieve our purpose, because any real closed field R is order-preserving isomorphic with a real closure of some field $\mathbb{Q}(\Lambda_T)$. We assume the Kuratowski-Zorn Lemma, so the set T can be well-ordered, provided $T \neq \emptyset$.

L. Bröcker [Bröc_1] proved in his Ultrafilter Theorem that there exists a one-to-one correspondence between the family of ultrafilters and the family of orderings in $\mathbb{Q}(\Lambda_T)$ or equivalently with the family of real closures of $\mathbb{Q}(\Lambda_T)$. In [32] we prove that there exists a one-to-one correspondence between the family of orderings in $\mathbb{Q}(\Lambda_T)$ and the family of plain filters [32, Theorem 5.2, Proposition 2.4, Corollary 2.5]. By a *plain filter* we mean a filter Ω of subsets of $\mathbb{R}^T$ defined by:

1) for any algebraic set $V \subsetneq \mathbb{R}^T$, where $V = P^{-1}(0)$, $P \in \mathbb{Q}[\Lambda_T]$, some connected component of $\mathbb{R}^T \setminus V$ belongs to Ω and any $U \in \Omega$ is of the above form,

2) for any $U_1, U_2 \in \Omega$ there exists $U_3 \in \Omega$ such that $U_3 \subset U_1 \cap U_2$.

The above mentioned correspondence between orderings and plain filters is as follows: for any ordering $\succ$ of $\mathbb{Q}(\Lambda_T)$ there exists a unique plain filter Ω such that $f \succ 0$ iff $f > 0$ on some $U \in \Omega$, where $>$ is the usual ordering on $\mathbb{R}$. Conversely, any plain filter Ω determines a unique ordering $\succ$ of $\mathbb{Q}(\Lambda_T)$ in the above way.

The main result of this article is [32, Theorem 5.2], where we give a construction of any real closure of $\mathbb{Q}(\Lambda_T)$ in terms of Nash functions. A function $f : U \to \mathbb{R}$ we call $\mathbb{Q}$-*Nash function*, if U is a topological component of some set $\mathbb{R}^T \setminus V$, where $V = P^{-1}(0)$, $P \in \mathbb{Q}[\Lambda_T]$ and there exists a nonzero polynomial $F \in \mathbb{Q}[\Lambda_T, Z]$ such that $F(\lambda, f(\lambda)) = 0$ for $\lambda \in U$. For any plain filter Ω and any $U \in \Omega$, the ring $\mathcal{N}(U)$ of $\mathbb{Q}$-Nash functions on U is a domain. In $\bigcup_{U\in\Omega} \mathcal{N}(U)$ we introduce an equivalence relation "$\sim$": $(f_1 : U_1 \to \mathbb{R}) \sim (f_2 : U_2 \to \mathbb{R})$ iff $f_1|_{U_3} = f_2|_{U_3}$ for some $U_3 \in \Omega$. Then the set $\mathcal{N}_\Omega$ of equivalence classes of "$\sim$" with the usual operations of addition and multiplication is a field, which is a real closure of $\mathbb{Q}(\Lambda_T)$.

In the language of plain filters we have the following characterization of Archimedean orderings in the field $\mathbb{Q}(\Lambda_T)$:

Theorem 32 ([32], Theorem 3.1) *An ordering $\succ$ of $\mathbb{Q}(\Lambda_T)$ is Archimedean iff for the plain filter Ω determining $\succ$, the set $\bigcap_{U\in\Omega} U$ is nonempty*

The above results refer to the geometric construction of the algebraic closure of a rational functions field $\mathbb{C}(\Lambda_1, \dots, \Lambda_m)$ obtained before the habilitation in [7].

In this model, each differentiation $d : \mathcal{N}_\Omega \to \mathcal{N}_\Omega$ is of the form $d(f) = \sum_{t\in T} g_t \frac{\partial f}{\partial x_t}$, where $(g_t \in \mathcal{N}_\Omega : t \in T)$ is an arbitrary system of elements of the set $\mathcal{N}_\Omega$. The construction of the field $\mathcal{N}_\Omega$ is transferred to any algebraically closed field of characteristic zero, and therefore the differentially closed fields must be of such a form. It is difficult to decide how to select the differentiation, to get the differentially closed field.

Research projects

In 2007 I supervised a PhD project no NN201 2600 33 entitled: *Łojasiewicz inequality and analytic equivalence of functions at infinity*, which resulted in the doctoral thesis by G. Skalski.

In 2009-2010, I was a Polish coordinator of the Grant POLONIUM no 7862/R09/R10 under the title: *Semialgebraic sets and mappings at infinity.* The coordinator of the French side was K. Kurdyka. The papers [27, 28] are results of the project. During the project, a concept of scientific cooperation between our Department and Laboratoire de Mathematiques de l' Universite de Savoie Mont Blanc (Francja) was developed.

Since 07/08/2013 I have ben the head of realization of the Polish National Science Centre, grant 2012/07/B/ST1/03293 under the title *Sums of squares and the Łojasiewicz exponent.* So far the result of this project is a series of scientific papers [33, 34, 36, 37, 38, 39, 40], and works by other authors. The results were presented at national and foregin scientific conferences.

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Achievements in the field of scientific supervision and education of young staff

Proceedings for granting the doctoral degree

Completed doctoral thesis for which the applicant was a supervisor

1. dr Skalski Grzegorz, University of Lodz, the year of obtaining the degree 2007.
Thesis title: *Łojasiewicz inequality and analytic equivalence of functions at infinity.*

2. dr Osińska-Ulrych Beata, University of Lodz, the year of obtaining the degree 2008.
Thesis title: *Extensions of regular mappings with preserving the Łojasiewicz exponent at infinity.*

3. dr Michalska Maria, University of Lodz, Uniwersyte Savoie Mont Blanc, the year of obtaining the degree 2012. Doctorate under cotutelle. On the French side of the University of Savoy, supervised by professor Krzysztof Kurdyka.
Thesis title: *Algebras of bounded polynomials on unbounded semialgebraic sets.*

4. dr Różycki Adam, University of Lodz, the year of obtaining the degree 2014.
Thesis title: *Effective characterization of a set of linear mappings defining multiplicity of a polynomial mapping at an improper zero.*

Open doctoral programs in which the applicant is a supervisor

1. mgr Migus Piotr, University of Lodz, the Ph.D. program open in 2013.,
Thesis title: *Local equivalence of class* C^r – completed doctoral thesis.

2. mgr Szlachcińska Anna, University of Lodz, the Ph.D. program open in 2013.,
Thesis title: *Łojasiewicz exponent of semialgebraic sets and mappings.*

3. mgr Klepczarek Michał, University of Lodz, the Ph.D. program open in 2014.,
Thesis title: *Trivialisation of analytic function on a hypersurface.*

PhD students for whom the applicant is an academic supervisor

1. mgr Gala-Jaskórzyńska Aleksandra, University of Lodz,

2. mgr Kuta Katarzyna, University of Lodz.

Prepared reviews in doctoral and habilitation proceedings

Participation in habilitation proceedings:

1. Jasiczak Michał, Adam Mickiewicz University in Poznan – Reviewer 2013 r.
Scientific work: *The problem of divisibility and interpolation for holomorphic functions of several variables.*

2. Białas-Cież Leokadia, Jagiellonian University - Reviewer 2014r.
Scientific work: *Selected polynomial inequalities in the context of the Green's functions.*

3. Kosiński Łukasz, Jagiellonian University - Reviewer 2016 r.
Scientific work: *Interpolation problems of Nevanlinna-Pick.*

Participation in the proceedings for awarding the degree of Doctor:

1. Brzostowski Szymon, University of Lodz - reviewer 2008 r.
Thesis title: *Approximative roots of polynomials.*

2. Kowalska Agnieszka, Jagiellonian University - reviewer 2008 r.
Thesis title:*Polynomial approximation on semialgebraic sets.*

3. Oleksik Grzegorz, University of Lodz - reviewer 2011 r.
Thesis title: *Łojasiewicz exponent of nondegenerate singularities.*

4. Antoniewicz Anna, Jagiellonian University - reviewer 2011 r.
Thesis title: *On some surface with a divisible sets of singularities.*

5. Walewska Justyna, University of Lodz - reviewer 2012 r.
Thesis title: *Milnor number of non-degenerate families of singularities of plane curves.*

Other achievements in the education of young staff

I have published on the website of Faculty of Mathematics and Computer Science, University of Lodz a textbook [49] for lectures on Mathematical Analysis 1 and 2 (416 pages). It includes within its scope lectures on one-dimensional Mathematical Analysis. The first version of this book [25] was published on the website of Faculty of Mathematics and Computer Science, University of Lodz before my habilitation. The current version is revised and expanded. Section on infinite products has been added; section on the Fourier series has been expanded; there has also been included a section on complex numbers, where Fundamental theorem of algebra and transcendence of the numbers π and e are proved. After each chapter students can also find sets of exercises.

The activities popularization of science

Within the framework of popularizing science, together with T. Krasinski we edited and published the selected papers of prof. Z. Charzyńaki (see [42] in the list of publications).

Together with a student A. Rogala, we published a series of student papers as post-conference materials (see [43]).

I organized four science camps for students of the Faculty of Mathematics and Computer Science, University of Lodz as a part of projects *Ordered Fields of Studies* in the Human Capital Operational Programme: Bukowina Tatrzańska, 06.07 - 15.07.2010 r.; Szklarska Poręba, 08.07 - 17.07.2011 r.; Szczyrk, 05.07 - 14.07.2012 r.; Szczyrk, 11.07 - 20.07.2013 r.

I gave lectures at conferences and scientific students camps:

1. Scientific students camps, Bukowina Tatrzańska 06.07-15.07.2010.
Lecture entitled: *On systems of polynomial equations and inequalities.*

2. Scientific students camps, Szczyrk 11.07-20.07.2013.
Lecture entitled: *Real fields.*

3. Conference Horizons in mathematics - WCMCS conference for students, Będlewo 17.03-21.03. 2014.
Lecture entitled:*Convex polynomials and sums of squares approximation.*

Organizational activity

Since 2010, I have been the head of the Department of Analytic Functions and Differential Equations at the Faculty of Mathematics and Computer Science, University of Lodz.

In terms 2008-2012 and 2012-2016 I held office as Deputy Dean for Student Affairs at the Department of Mathematics and Computer Science, University of Lodz.

Since 2010, I have been an organizer of the annual national conferences: Conference and Workshop "Analytic and Algebraic Geometry" organized at the University of Lodz. 37 have been held up till now.

I established a direct scientific cooperation with the University of Savoie Mont Blanc (France). The agreement was signed in 2008. As part of the cooperation, we promoted a doctorate in the system of cotutelle and gave a series of research papers on the metric properties of semialgebraic sets and applications in optimization.

I was in charge of three projects of Human Capital Operational Programme in 2008-2015 in the framework of the so-called Ordered Fields of Studies.