On the weighted bounded negativity conjecture for algebraic surfaces

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Analytic and Algebaric Geometry Conference
Łódź, January 6-10, 2020

Definition

We say that a surface X has the *Bounded Negativity Property* if there exists a number $b(X)$ such that

$$C^2 \geq -b(X)$$

holds for all reduced and irreducible curves $C \subset X$.

The Bounded Negativity Property

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Example

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Example

- For $\mathbb{P}^2$ it suffices to take $b(\mathbb{P}^2) = 0$.
- For the Hirzebruch surface $\mathbb{F}_n$, $b(\mathbb{F}_n) = n$ suffices.

Conjecture

Every **complex** *surface has the Bounded Negativity Property.*

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Remark

This conjecture **fails** in positive characteristic!

Conjecture (WBNC)

For every smooth projective surface X over the complex numbers, there exists a non-negative integer $b_w \in \mathbb{Z}$ such that $C^2 \geq -b_w(X) \cdot (C.H)^2$ for all integral curves $C \subset X$ and all big and nef line bundles H for which $C.H > 0$.

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Remark

This problem is really widely open.

Remark

Notice that we are not asking for the self-intersection number of a curve C *to be bounded from below, but rather that the weighted self-intersection* $C^2/(C.H)^2$ *of* C *be so, hence the adjective "weighted". Put differently, WBNC is asking for a bound on the self-intersection of all integral curves on* X *that depends on both* X *and the degree of the curve* C *with respect of every big and nef line bundle over which the curve is positive.*

Here we provide bounds for the self-intersection numbers of irreducible and reduced curves on blow-ups of algebraic surfaces at mutually distinct points. The bounds depend on the degree of the curve with respect to an explicitly constructed big and nef line bundle Γ, and in fact it holds for the cone $\mathrm{Nef}(X) + \Gamma$.

Theorem (Orevkov-Zaidenberg '95, Sakai '90)

Let C be a reduced and irreducible curve in $\mathbb{P}^2$ of degree d having singular points $p_1, ..., p_s$. We denote by m_{p_i} and μ_{p_i} the corresponding multiplicity and the Milnor number of p_i. If the logarithmic Kodaira dimension of $\mathbb{P}^2 \setminus C$ is non-negative, then

$$\sum_{i=1}^{s} \left(1 + \frac{1}{2m_{p_i}}\right)\mu_{p_i} \leq d^2 - \frac{3}{2}d.$$

Wakabayashi's result

As it was shown by Wakabayashi [4], if $D \subset \mathbb{P}^2$ is an irreducible and reduced curve of degree $d \geq 4$ having $s \geq 1$ singular points, which is not a rational cuspidal curve with one cusp, then the logarithmic Kodaira dimension of $\mathbb{P}^2 \setminus D$ is non-negative.

Theorem

Let $\sigma : Y \longrightarrow \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at $S = \{p_1, \ldots, p_n\}$, where the p_i's are distinct points of $\mathbb{P}^2$, and let C be an irreducible and reduced curve on Y. Then,

$$C^2 \geq -2n(C.L),$$

where L is the pull-back of a line in $\mathbb{P}^2$.

Remark

We can do better using Plücker-Teissier formula, and then we get

$$C^2 \geq -n(C.L)$$

Moreover, if the points are very general, then by Ein-Lazarsfeld-Xu lemma we get

$$-C^2 \leq \min\{\mathrm{mult}_{P_i} C \neq 0\}.$$

Theorem

Let C be an irreducible and reduced curve in a smooth complex projective surface X having singular points $p_1, ..., p_s$. We denote by m_{p_i} and μ_{p_i} the corresponding multiplicities and the Milnor numbers of p_i's. Assume that the logarithmic Kodaira dimension of $X \setminus C$ is non-negative, then one has

$$\sum_{p \in Sing(C)} \left(2 + \frac{1}{m_p}\right)\mu_p \leq 3e(X) - K_X^2 + 2C^2 + K_X.C.$$

Let X a smooth projective surface over the complex numbers, and let $\sigma : Y \longrightarrow X$ be the blow-up of X at $S = \{p_1, \ldots, p_n\}$, where the p_i's are mutually distinct points of X.

Theorem

There exists an ample line bundle $\Delta \in \mathrm{Pic}(X)$ (which is of the form $K_X + 3nA$ for suitable chosen A very ample) such that

$$C^2 \geq -\frac{1}{2}\big(\delta(X) + (\Delta.\overline{C})\big) - n,$$

for all integral curves $C \subset Y$ such that the logarithmic Kodaira dimension $X \setminus \overline{C}$ is non-negative. Here, $\overline{C} := \sigma(C)$, $\delta(X) := 3e(X) - K_X^2$ is the Miyaoka-Yau number.

Theorem

Assume X is a surface of non-negative Kodaira dimension. Then, in the setting above, there exists a big and nef line bundle Γ that bounds negativity, i.e.,

$$C^2 \geq -\frac{1}{2}\left(\delta(X) + C.\Gamma\right) - n,$$

for every integral curve $C \subset Y$. In other words, if we define $\deg_\Gamma C := (C.\Gamma)$, *then*

$$C^2 \geq -\left(\frac{1}{2}\delta(X) + n\right) - \frac{1}{2}\deg_\Gamma C,$$

i.e., the negativity of C is bounded by a function that depends on X, the number of points we have blown-up, and the Γ-degree of C.

Problems:

- How to construct irreducible and reduced curves in the complex projective plane having very low self-intersection numbers?
- Does there exist an irreducible and reduced plane curve having degree 11 and 15 triple intersection points?
- Is there a way to produce families of irreducible and reduced plane curves having a large number of triple / quadruple points?

References

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Thank you for your attention.