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PLANE ALGEBROID BRANCHES
AFTER R. APÉRY

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The aim of these notes is to present the approach to plane algebroid branches proposed by Roger Apéry in his 1946 note [A] and subsequently developed by Azevedo [Az] and Angermüller [An]. In what follows we use freely the notions and theorems explained in [Pł].

The essence of Apéry method is as follows. Let us consider an algebroid branch given by a good parametrization $x = \varphi(t)$, $y = \psi(t)$. Then the integers $\operatorname{ord} g(\varphi(t), \psi(t))$, where $g = g(x, y)$ runs over formal power series not vanishing on the branch form the semigroup G associated with the branch. To study the semigroup G Apéry introduced the important notions presently called Apéry sequence and Apéry basis. With these notions he was able to study the relationship between the semigroups associated with the given branch and its strict quadratic transformation. Then he proved the main result: G is a symmetric semigroup i.e. there is an integer $A > 0$ such that for any integers a, b: if $a + b = A$ then exactly one element of the pair a, b belongs to G. An important application of this property is the theorem: the local ring of a plane algebroid branch is a Gorenstein ring (see [G]).

The following quotation from [A] shows that Apéry knew Gorenstein's property as early as in 1946:

> "On peut poser $A = 2P - 1$ (A est impair d'après ses propriétés) et appeler P le genre de la branche de courbe. P est la quantité dont

> augumenterait le genre de la courbe en considérant les singularités dues à la branche comme virtuellement inexistantes. Le nombre de points absorbé par l'intersection de la branche avec une adjointe est $2P$. Le nombre de conditions imposées par la branche aux adjointes est P."

The proof of Gorenstein's property presented in these notes is due to Azevedo [Az].

1. Numerical semigroups

A *numerical semigroup* G is a subset of $\mathbb{N}$ closed under addition, containing 0 and such that the set $\mathbb{N} \setminus G$ is finite. The minimal element $c \in \mathbb{N}$ such that $c + \mathbb{N} \subset G$ is called the conductor of G. The number $c - 1$ is then the biggest integer not belonging to G and it is called the Frobenius number of G.

Lemma 1.1. *Let $a, b > 0$ be coprime integers. Then $G = a\mathbb{N} + b\mathbb{N}$ is a numerical semigroup whose conductor is $c = (a-1)(b-1)$.*

Proof. Let us start with the following claim:

(*) for any $n \in \mathbb{Z}$ there is a unique pair $(p, q) \in \mathbb{Z}^2$ such that $n = pa + qb$ and $0 \le q < a$.

Uniqueness: if $pa + qb = p'a + q'b$ with $0 \le q \le q' < a$ then a divides $(q' - q)b$ and consequently $q' - q$ since a, b are coprime. Therefore $q' - q = 0$ and obviously $p' - p = 0$.

Existence: there exist integers P, Q such that $n = Pa + Qb$. For any l we have $Pa + Qb = (P - lb)a + (Q + la)b$. Choose $l \in \mathbb{Z}$ such that $0 \le Q + la < a$ and take $p = P - lb$, $q = Q + la$ ■

Now, we can prove Lemma 1.1. Let $n \ge (a-1)(b-1)$ and let $n = pa + qb$ with p, q such that in (*). Then $pa = n - qb \ge (a-1)(b-1) - (a-1)b = -a + 1$ and $p \ge -1 + \frac{1}{a}$. Therefore $p \ge 0$ and $n \in G$ since $q \ge 0$. Thus G is a numerical semigroup and its conductor is less that or equal to $(a-1)(b-1)$.

To prove the equality $c = (a-1)(b-1)$ we have to check that $(a-1)(b-1) - 1 \notin G$. Suppose to the contrary that $(a-1)(b-1) - 1 = Ra + Sb$ with $R, S \in \mathbb{N}$. Dividing S by a with the rest $Q < a$, $Q \ge 0$ we get $Ra + Sb = Ra + (S_1 a + Q)b = (R + S_1 b)a + Qb = Pa + Qb$, where $P = R + S_1 b \in \mathbb{N}$. On the other hand $(a-1)(b-1) - 1 = (-1)a + (a-1)b$ and by the uniqueness of the representation (*) we get $P = -1$. A contradiction ■

A numerical semigroup G is symmetric if there exists an integer $A \in \mathbb{N}$ such that for any $a, b \in \mathbb{Z}$: if $a + b = A$ then exactly one element of the pair a, b belongs to G. Note that $A \notin G$ $(0 + A = A$ and $0 \in G)$ and any integer $n \ge A + 1$ belongs to G $(n + (A - n) = A$, $A - n < 0$ and $A - n \notin G)$. If c is the conductor of G then $c = A + 1$.

Lemma 1.2. *Let G be a symmetric semigroup with the conductor c. Then for any $a \in \mathbb{Z}$: $a \in G \Leftrightarrow c - 1 - a \notin G$. Moreover, c is an even number, $\#(\mathbb{N} \setminus G) = \#(G \cap [0, c-1]) = c/2$.*

Proof. We have $a + ((c-1) - a) = c - 1$ and $c - 1 = A$. Therefore we have that $a \in G \Leftrightarrow c - 1 - a \notin G$. To check the second part of the lemma observe that the mapping

$$[0, c-1] \cap G \ni z \to [0, c-1] \cap (\mathbb{N} \setminus G)$$

is a bijection ∎

Let G be a numerical semigroup and let $n \in G \setminus (0)$. The *Apéry sequence* of G with respect to n is, by definition, the sequence $a_0, \ldots, a_{n-1} \in G$ which satisfies the following three properties:

1. $0 = a_0 < a_1 < \cdots < a_{n-1}$,
2. $a_i \neq a_j \pmod{n}$ for $i \neq j$,
3. if $a \in G$ and $a \geq a_i$.

It is easy to check that G and n define exactly one Apéry sequence (pick up in each residue class od $\mathbb{Z}$ mod n the smallest element belonging to G, then write up these elements in their natural order).

Let $\mathrm{Ap}(G, n) = (a_0, a_1, \ldots, a_{n-1})$. For any integer $a \in \mathbb{Z}$ there are unique $i \in \{0, \ldots, n-1\}$ and $l \in \mathbb{Z}$ such that $a = a_i + ln$. Obviously, we have $a \in G$ if and only if $l \geq 0$. Note also that

$$G = \bigcup_{i=0}^{n-1} (a_i + n\mathbb{N}) \; .$$

Lemma 1.3. *Let c be the conductor of G. Then $c = a_{n-1} - n + 1$.*

Proof. Observe that $a_{n-1} - n = a_{n-1} + (-1)n \notin G$ and if $a \notin G$ then $a = a_i + ln$ with $l < 0$ which implies $a = a_i + ln \leq a_{n-1} - n$. Therefore $a_{n-1} - n$ is the Frobenius number of G and $a_{n-1} - n = c - 1$ ∎

Lemma 1.4. *Let $\mathrm{Ap}(G, n) = (a_0, \ldots, a_{n-1})$. Then the following two conditions are equivalent*

(i) *the semigroup G is symmetric,*
(ii) *$a_i + a_{n-1-i} = a_{n-1}$ for $i = 0, 1, \ldots, n-1$.*

Proof

(ii) $\Rightarrow$ (i). Let $a, b \in \mathbb{Z}$ be such that $a + b = a_{n-1} - n$. Suppose that $a \in G$. We have $a = a_i + ln$, where $i \in \{0, \ldots, n-1\}$ and $l \in \mathbb{N}$. Therefore $b = a_{n-1} - a_i - ln - n = (a_{n-1} - a_i) - (l+1)n = a_{n-1-i} - (l+1)n$. Since $-(l+1) < 0$ we get $b \notin G$.

(i) $\Rightarrow$ (ii). First we check that for any $i \in \{0, \ldots, n-1\}$ there exists a $j \in \{0, \ldots, n-1\}$ such that $a_i + a_j = a_{n-1}$. Fix $i \in \{0, \ldots, n-1\}$ and let $b = a_{n-1} - n - a_i$. Since $a_i + b = a_{n-1} - n$ and G is symmetric we get $b \notin G$ and consequently $b = a_j + ln$ with $l < 0$. We have $a_i + (a_j + ln) = a_{n-1} - n$, that is $(a_i - n) + (a_j + (l+1)n) = a_{n-1} - n$. By symmetry of G we get $a_j + (l+1)n \in G$ ($a_i - n \notin G$) which implies $l + 1 \geq 0$. Therefore $l = -1$ and $b = a_j - n$. Consequently $a_i + (a_j - n) = a_{n-1} - n$ i.e. $a_i + a_j = a_{n-1}$. By the above property for any $i \in \{0, \ldots, n-1\}$ there exists a $j(i) \in \{0, \ldots, n-1\}$ such that $a_i + a_{j(i)} = a_{n-1}$. The sequence a_i is increasing, therefore the sequence $a_{j(i)}$ is decreasing which implies that $j(i) = n - 1 - i$ for all $i \in \{0, \dots, n-1\}$ ■

Example 1.5. *If* $G = \mathbb{N}\, a + \mathbb{N}\, b$, *where* $\gcd(a, b) = 1$ *then* $\mathrm{Ap}(G, a) = (0, b, \dots, (a-1)b)$ *and* G *is symmetric.*

2. The value semigroup of a plane algebroid branch

We recall here some basic notions from the local theory of plane algebroid curves. For more details we refer the reader to [Pł]. In what follows $\mathbb{K}$ is an algebraically closed field of arbitrary characteristic. Let $\mathbb{K}[[x, y]]$ be the ring of formal power series in two variables x, y with coefficients in $\mathbb{K}$. The plane algebroid branch is by definition the principal prime ideal of $\mathbb{K}[[x, y]]$. The branch $(f)\mathbb{K}[[x, y]]$ where f is an irreducible power series will be denoted $\{f = 0\}$. For any branch $\{f = 0\}$ there exists a pair $(\varphi(t), \psi(t))$ of power series without constant term, in one variable t such that $f(\varphi(t), \psi(t)) = 0$ in $\mathbb{K}[[t]]$ and $\mathbb{K}((\varphi(t), \psi(t))) = \mathbb{K}((t))$ (see [Pł], Section 2). We call the pair $(\varphi(t), \psi(t))$ the normalization of the branch $\{f = 0\}$. For every $g = g(x, y) \in \mathbb{K}[[x, y]]$ we define:

$$\nu_f(g) = \operatorname{ord} g(\varphi(t), \psi(t)) \in \mathbb{N} \cup \{+\infty\}\ .$$

Let us recall the basic properties of ν_f (see [Pł], Section 3):

(i) $\nu_f(g) = 0$ if and only if $g(0) \neq 0$,
$\nu_f(g) = +\infty$ if and only if f divides g.
(ii) $\nu_f(g + g') \geq \inf\{\nu_f(g), \nu_f(g')\}$ with equality if $\nu_f(g) \neq \nu_f(g')$.
(iii) $\nu_f(gg') = \nu_f(g) + \nu_f(g')$,
(iv) if $\nu_f(g) = \nu_f(h) < +\infty$ then there exists a constant $c \in \mathbb{K}$ such that $\nu_f(g - ch) > \nu_f(g)$.

For any irreducible power series $f \in \mathbb{K}[[x, y]]$ we put

$$G(f) = \{\nu_f(g) : g \text{ runs over all power series such that } g \not\equiv 0 \,(\mathrm{mod}\, f)\}\ .$$

Clearly $G(f) \subset \mathbb{N}$ is a semigroup. We call $G(f)$ the *semigroup associated with the branch* $\{f = 0\}$. Two branches $\{f = 0\}$ and $\{g = 0\}$ are *equisingular* if and only if $G(f) = G(g)$. The branch $\{f = 0\}$ is nonsingular (i.e. of multiplicity 1) if and only if $G(f) = \mathbb{N}$. We have $\min(G(f) \setminus \{0\}) = \operatorname{ord} f$.

Property 2.1. *The semigroup* $G(f)$ *is numerical.*

Proof. Let $(\varphi(t), \psi(t))$ be the normalization of the branch $\{f = 0\}$. Then we have $\mathbb{K}((t)) = \mathbb{K}((\varphi(t), \psi(t))$ and we can write $t = \dfrac{p(\varphi(t), \psi(t))}{q(\varphi(t), \psi(t))}$ for some $p(x, y), q(x, y) \in \mathbb{K}[[x, y]]$, $q \not\equiv 0 \,(\mathrm{mod}\, f)$. Taking orders give $1 = \nu_f(p) - \nu_f(q)$. Put $a = \nu_f(p)$ and $b = \nu_f(q)$. Then $a, b \in G(f)$ are coprime and $G(f)$ is numerical by Lemma 1.1 ■

Example 2.2. Let $f = x^a + y^b + \sum c_{\alpha\beta} x^\alpha y^\beta$, where $\frac{\alpha}{a} + \frac{\beta}{b} > 1$, $\gcd(a, b) = 1$. Then f is irreducible and $G(f) = \mathbb{N}\, a + \mathbb{N}\, b$.

In what follows we write $\mathrm{Ap}(f, n)$ instead of $\mathrm{Ap}(G(f), n)$.

3. Apéry bases

Let us begin with some notations. If R is a commutative ring with $1 \neq 0$, we shall denote by $R[y]_{\mathrm{mon}}$ the semigroup of monic polynomials with coefficients in R, in one variable y. For any integer $k > 0$ we let $R[y]^{(k)} := \{h \in R[y] : \deg_y R < k\}$ (by convention $\deg_y 0 = -\infty$). If $h_0 = 1$, $h_1, \ldots, h_{k-1} \in R[y]_{\mathrm{mon}}$, $\deg_y h_i = i$ for $i = 0, 1, \ldots, k-1$ then $R[y]^{(k)} = \sum_{i=0}^{k-1} R\, h_i(y)$.

Let $f \in \mathbb{K}[[x,y]]$ be an irreducible power series. Assume that $n := \nu_f(x) < +\infty$ and let $\mathrm{Ap}(f,n) = (a_0, a_1, \ldots, a_n)$.

Theorem 3.1. *There exists a sequence of polynomials* $h_0, \ldots, h_{k-1} \in \mathbb{K}[[x]][y]_{\mathrm{mon}}$ *such that* $\deg_y h_k = k$ *and* $\nu_f(h_k) = a_k$ *for* $k = 0, 1, \ldots, n-1$.

We call the sequence of polynomials such as in the theorem above an Apéry basis of $\mathbb{K}[[x,y]]/(f)$ with respect to x. Note that

$$\mathbb{K}[[x,y]]\Big/(f) \cong \sum_{i=0}^{n-1} \mathbb{K}[[x]]\, h_i \ .$$

The proof of Theorem 3.1 is based on two lemmas.

Lemma 3.2. *Let* $k \in \{1, \ldots, n-1\}$ *and suppose that there exists a sequence* $h_0, \ldots, h_{k-1} \in \mathbb{K}[[x]][y]_{\mathrm{mon}}$ *such that* $\deg_y h_i = i$ *and* $\nu_f(h_i) = a_i$ *for* $i = 0, 1, \ldots, k-1$. *Then*

$$\{\nu_f(g) : g \in \mathbb{K}[[x]][y]^{(k)} \setminus \{0\}\} = \bigcup_{i=0}^{k-1} (a_i + \mathbb{N}\, n) \ .$$

Proof of Lemma 3.2

If $a \in \bigcup_{i=0}^{n-1}(a_i + \mathbb{N}\, n)$ then $a = a_i + l\, n$ for an $i \in \{0, \ldots, k-1\}$ and $l \geq 0$. Let $g = x^l h_i$. Then $\deg_y g = \deg_y h_i = i \leq k-1$ and $\nu_f(g) = l\, n + a_i = a$ that is $a \in \{\nu_f(g) : g \in \mathbb{K}[[x]][y]^{(k)} \setminus \{0\}\}$. Fix a polynomial $g \in \mathbb{K}[[x]][y]$ such that $g \notin 0$ and $\deg_y g \leq k-1$. Then we can write $g = A_{k-1}h_{k-1} + \cdots + A_0 h_0$ where $A_i \in \mathbb{K}[[x]]$ for $i = 0, 1, \ldots, k-1$. Let $I = \{i \in [0, k-1] : A_i \neq 0\}$. For any $i \in I$ we have $\nu_f(A_i h_i) = (\mathrm{ord}\, A_i)\, n + a_i$. Therefore $\nu_f(A_i h_i) \neq \nu_f(A_j h_j)$ for $i \neq j$ and we get $\nu_f(g) = \inf\{\nu_f(A_i h_i)\} = \nu_f(A_{i_0} h_{i_0}) \in a_{i_0} + n\, \mathbb{N}$ for an $i_0 \in I$. Consequently we obtain $\nu_f(g) \in \bigcup_{i=0}^{k-1}(a_i + \mathbb{N}\, n)$ ■

Lemma 3.3. *Under the assumptions of Lemma 3.2:*

(a) *if* $g \in \mathbb{K}[[x]][y]^{(k)}$ *then* $\nu_f(g + y^k) \leq a_k$,

(b) *if* $\nu_f(y^k + g) \in \bigcup_{i=0}^{k-1}(a_i + \mathbb{N}\, n)$ *for a polynomial* $g \in \mathbb{K}[[x]][y]^{(k)}$ *then there exists a polynomial* $\bar{g} \in \mathbb{K}[[x]][y]^{(k)}$ *such that* $\nu_j(y^k + g) < \nu_j(y^k + \bar{g})$.

Proof of Lemma 3.3

(a) Let $g \in \mathbb{K}[[x]][y]^{(k)}$. Since $a_k \in G(f)$ there exists a polynomial $h \in \mathbb{K}[[x]]^{(n-1)}$ such that $\nu_f(h) = a_k$. By the Euklidean division we get $h = (y^k + g)Q + R$, where $\deg_y R \leq k-1$ (it follows from Lemma 3.2 that $k \leq \deg_y h$ since $a_k \notin \bigcup_{i=0}^{k-1}(a_i + \mathbb{N}\, n)$). Clearly $\nu_f(h) \neq \nu_f(R)$ and we get $a_k \geq \nu_f(h - R) = \nu_f((y^k + g)Q) \geq \nu_f(y^k + g)$ which proves (a).

(b) If $\nu_f(y^k+g) \in \bigcup_{i=0}^{k-1}(a_i+\mathbb{N}\,n)$ then by Lemma 3.2 there exists a polynomial $g_1 \in \mathbb{K}[[x]][y]^{(k-1)}$ such that $\nu_f(y^k+g) = \nu_f(g_1)$. Therefore there is a constant $c \in \mathbb{K}$ such that $\nu_f(y^k+g-cg_1) > \nu_f(g_1) = \nu_f(y^k+g)$. It suffices to take $\bar{g} = g - cg_1$ ∎

Proof of Theorem 3.1
Let $k \in \{1,\dots,n\}$ be the greatest integer such that there exists a sequence $h_0,\dots,$ $h_{k-1} \in \mathbb{K}[[x]]_{\mathrm{mon}}$ satisfying the conditions $\deg_y h_i = i$, $\nu_f(h_i) = a_i$ for $i = 0,1,\dots,k-1$. Such a sequence exists for $k=1$ (take $h_0 \equiv 1$). We claim that $k=n$. Suppose to the contrary that $k<n$ and consider the set of polynomials y^k+g, $\deg_y g \le k-1$. By Lemma 3.3 (a) we have $\nu_f(y^k+g) \le a_k$ so that the set $\nu_f(y^k+g) : g \in \mathbb{K}[[x]][y]^{(k)} \setminus \{0\}\}$ is finite. By Lemma 3.3 (b) there exists a polynomial $\bar{g}$, $\deg_y \bar{g} \le k-1$ such that $\nu_f(y^k+g) \notin \bigcup_{i=0}^{n-1}(a_i+\mathbb{N}\,n)$. Let $h_k = y^k+\bar{g}$. Thus $\nu_f(h_k) \equiv a_j \pmod{n}$ for $j \ge k$ and $\nu_f(h_k) \ge a_j \ge a_k$. From Lemma 3.3 (a) we infer that $\nu_f(h_k) = a_k$. Since $\deg_y h_k = k$ we obtained a contradiction which proves that $k=n$ ∎

Corollary 3.4. *Let $h_0,\dots,h_{k-1}$ be an Apéry basis of $\mathbb{K}[[x,y]]/(f)$ with respect to x. Then for any $k \in \{1,\dots,n\}$:*

(a) $\{\nu_f(g) : g \in \mathbb{K}[[x]][y]^{(k)} \setminus \{0\}\} = \displaystyle\bigcup_{i=0}^{k-1}(a_i+\mathbb{N}\,n).$

(b) *If $g \in \mathbb{K}[[x]][y]^{(k)}$ then $\nu_f(y^k+g) \le a_k$. The equality $\nu_f(y^k+g) = a_k$ holds if and only if $\nu_f(y^k+g) \notin \bigcup_{i=0}^{n-1}(a_i+\mathbb{N}\,n)$.*

Proof. Part (a) follows from Lemma 3.2, part (b) from Lemma 3.3 (a) ∎

The following characterization of Apéry bases will be useful.

Theorem 3.5. *Suppose that there exists a sequence $h_0,\dots,h_{k-1} \in \mathbb{K}[[x]][y]_{\mathrm{mon}}$ such that $\deg_y h_k = k$, $\nu_f(h_k) \not\equiv \nu_f(h_l) \pmod{n}$ for $k \ne l$. Then $\mathrm{Ap}(f,n) = (\nu_f(h_0),\nu_f(h_1),\dots,\nu_f(h_{k-1}))$.*

Proof. Let $\mathrm{Ap}(f,n) = (a_0,a_1,\dots,a_{n-1})$ and let $k \in \{1,\dots,n\}$ be the greatest integer k such that $\nu_f(h_0) = a_0,\dots,\nu_f(h_{k-1}) = a_{k-1}$. We claim that $k=n$. Suppose to the contrary that $k<n$ and consider $\nu_f(h_k)$. We have $\nu_f(h_k) \le a_k$ by Lemma 3.3 (a). Moreover, $\nu_f(h_k) \not\equiv a_i \pmod{n}$ for $i \le k-1$. Therefore $\nu_f(h_k) \equiv a_j \pmod{n}$ for an integer $j \ge k$ and $\nu_f(h_k) \ge a_j \ge a_k$. Thus we get $\nu_f(h_k) = a_k$ which is a contradiction with the definition of k. Therefore we get $k=n$ ∎

Remark 3.6. The Apéry sequence $\mathrm{Ap}(f,n) = (a_0,\dots,a_{n-1})$ is *strongly increasing* i.e. $a_i + a_j \le a_{i+j}$ for $i+j \le n-1$. Indeed, $a_i = \nu_f(y^i+g_i)$, $\deg_y g_i \le i-1$ and $a_j = \nu_f(y^j+g_j)$, $\deg_y g_j \le j-1$. Therefore $a_i+a_j = \nu_f((y^i+g_i)(y^j+g_j)) = \nu_f(y^{i+j}+y^ig_j+y^jg_i+g_ig_j) \le a_{i+j}$ by Corollary 3.4 ∎

4. Quadratic transformations and multiplicity sequences

Let $f \in \mathbb{K}[[x,y]]$ be an irreducible power series. Then the branch $\{f=0\}$ has exactly one tangent (see [Pł], Property 1.1). Let $m = \operatorname{ord} f$ be the multiplicity of the branch $\{f=0\}$. We define the strict quadratic transformation $Q(f)$ of f as follows: if the unique tangent to $\{f=0\}$ has the equation $y - ax = 0$ the $Q(f) = f_1$ where $f_1 \in \mathbb{K}[[x_1,y_1]]$ is a series uniquely determined by the condition $f(x_1, ax_1 + x_1y_1) = x_1^m f(x_1,y_1)$, if the unique tangent to $\{f=0\}$ has the equation $x=0$ then $Q(f) = f_1$ where $f(x_1y_1, y_1) = x_1^m f_1(x_1,y_1)$ in $\mathbb{K}[[x_1,y_1]]$. The strict quadratic transformation $Q(f)$ is an irreducible power series and $\operatorname{ord} Q(f) \le \operatorname{ord} f$ (see [Pł], Lemmas 1.3 and 1.4).

The following theorem is due to Apéry ([A]).

Theorem 4.1. *Let $\{f=0\}$ be a branch of multiplicity m and let $Ap(f,m) = (a_0,\dots,a_{m-1})$ and $\mathrm{Ap}(Q(f),m) = (a'_0,\dots,a'_{m-1})$. Then $a_k = a'_k + km$ for $k = 0,1,\dots,m-1$.*

Proof. We may assume without loss of generality then the unique tangent to the branch $\{f=0\}$ is the line $y=0$. Then $Q(f) = f_1$ where the power series $f_1 \in \mathbb{K}[[x_1,y_1]]$ is determined by the condition $f(x_1,x_1y_1) = x_1^m f_1(x_1,y_1)$. Let

$$\tilde{h}_k = y_1^k + a_{k,1}(x_1)y_1^{k-1} + \cdots + a_{k,k}(x_1) \in \mathbb{K}[[x_1]][y_1], \text{ for } k = 0,1,\dots,m-1$$

be an Apéry basis of the branch $\{f_1 = 0\}$ with respect to x_1. Let

$$h_k = y^k + x\,a_{k,1}(x)y^{k-1} + \cdots + x^k a_{k,k}(x) \in \mathbb{K}[[x]][y], \text{ for } k = 0,1,\dots,m-1\,.$$

Obviously, we have $\deg_y h_k = k$ for $k = 0,1,\dots,m-1$. To compute $\nu_f(h_k)$ consider a good parametrization $(\varphi(t),\psi(t))$ of the branch $f=0$. Then $\operatorname{ord}\varphi(t) < \operatorname{ord}\psi(t)$ and the pair $(\varphi_1(t),\psi_1(t)) = (\varphi(t),\psi(t)/\varphi(t))$ is a good parametrization of the branch $f_1 = 0$. Therefore $\nu_f(h_k) = \operatorname{ord} h_k(\varphi(t),\psi(t)) = \operatorname{ord} h_k(\varphi_1(t),\varphi_1(t)\psi_1(t)) = \operatorname{ord}\varphi_1(t)^k \tilde{h}_k(\varphi_1(t),\psi_1(t)) = km + \operatorname{ord}\tilde{h}_k(\varphi_1(t),\psi_1(t)) = km + \nu_{f_1}(\tilde{h}_k) = km + a'_k$ for $k = 0,1,\dots,m-1$. Thus $\nu_f(h_k) \not\equiv \nu_f(h_l) \pmod{n}$ for $k \ne l$ and by Theorem 3.5 we get $(\nu_f(h_0),\dots,\nu_f(h_{m-1})) = (a_0,\dots,a_{m-1})$ which proves that $a_k = km + a'_k$ for $k = 0,\dots,m-1$ ■

Corollary 4.2. *The strict quadratic transformations of equisingular branches are equisingular.*

Corollary 4.3. *Let $\{f=0\}$ be a branch of multiplicity $m = \operatorname{ord} f$. If c and c_1 are respectively the conductors of $G(f)$ and $G(Q(f))$ then $c = c_1 + m(m-1)$.*

Proof. Use Theorem 4.1 and the formula for the conductor in terms of Apéry sequence (see Lemma 1.3) ■

For any branch $\{f=0\}$ we define $Q^k(f) = Q(\dots Q(f))$, where Q is repeated k times. We put $m_0 = \operatorname{ord} f$ and $m_k = \operatorname{ord} Q^k(f)$ for $k \ge 1$. The sequence of positive integers $(m_0, m_1, \dots, m_k, \dots)$ is called the multiplicity sequence of the branch $\{f=0\}$.

Theorem 4.4.

(4.4.1) *The multiplicity sequence is decreasing, all but finite number of its terms are equal to 1.*

(4.4.2) *Two branches are equisingular if and only if they have the same multiplicity sequence.*

(4.4.3) *If c is the conductor of the semigroup of a branch with multiplicity sequence $(m_0, m_1, \dots, m_k, \dots)$ then*

$$c = \sum_{k \geq 0} m_k(m_k - 1) .$$

Proof. The theorem easily follows from Corollaries 4.2 and 4.3.

Note that an arithmetical characterization of multiplicity sequences was given by Flenner and Zaindenberg ([F-Z], Proposition 1.2).

Theorem 4.5. (Apéry). *The semigroup of a plane branch is symmetric.*

Proof. Using the characterization of symmetric semigroups (Lemma 1.4) in terms of Apéry sequence we check using Theorem 4.1 that for any branch $\{f = 0\}$: $G(f)$ is symmetric if and only if $G(Q(f))$ is symmetric. If $\{f = 0\}$ is nonsingular then $G(f) = \mathbb{N}$ is obviously symmetric. Hence follows the theorem ∎

5. The Gorenstein theorem

Let $f \in \mathbb{K}[[x, y]]$ be an irreducible power series. We put

$$\begin{aligned} \mathcal{O}_f &= \mathbb{K}[[x,y]]/(f) , \\ \mathcal{M}_f &= \text{the field of fractions of } \mathcal{O}_f , \\ \hat{\mathcal{O}}_f &= \text{the normalization (=the integral closure) of } \mathcal{O}_f \text{ in } \mathcal{M}_f . \end{aligned}$$

Let $\mathcal{C}$ be the conductor of $\mathcal{O}_f$ i.e. the largest ideal in $\mathcal{O}_f$ which remains an ideal in $\hat{\mathcal{O}}_f$. Our aim is to prove the following result due to Gorenstein.

Theorem 5.1. $\dim_{\mathbb{K}} \hat{\mathcal{O}}_f/\mathcal{C} = 2 \dim_{\mathbb{K}} \mathcal{O}_f/\mathcal{C}$.

To deduce Theorem 5.1 from Apéry's property of the semigroup associated with a branch we need two lemmas. Let c be the conductor of the semigroup $G(f)$.

Lemma 5.2. $\dim_{\mathbb{K}} \hat{\mathcal{O}}_f/\mathcal{C} = c$.

Proof. Let $(\varphi(t), \psi(t))$ be a good parametrization of the branch $f = 0$. Then $\mathcal{O} = \mathbb{K}[[\varphi(t), \psi(t)]]$, $\hat{\mathcal{O}} = \mathbb{K}[[t]]$, $\mathcal{C} = \{d(t) \in \mathbb{K}[[t]] : d(t)\mathbb{K}[[t]] \subset \mathbb{K}[[\varphi(t), \psi(t)]]\}$ and $G(f) = \{\operatorname{ord} g(\varphi(t), \psi(t)) : g(\varphi(t), \psi(t)) \neq 0\}$. By [Pł] (Theorem 2.10) $\mathcal{C}$ is a nonzero ideal of the ring $\mathbb{K}[[t]]$. Therefore $N = \dim_{\mathbb{K}} \mathbb{K}[[t]]/\mathcal{C} < +\infty$ and $\mathcal{C} = (t^N)\mathbb{K}[[t]]$. By the definition of the conductor:

$\cdot\ t^N \mathbb{K}[[t]] \subset \mathbb{K}[[\varphi(t), \psi(t)]]$

and

$\cdot\cdot\ t^{N-1} \mathbb{K}[[t]] \not\subset \mathbb{K}[[\varphi(t), \psi(t)]]$.

From the above properties it follows that $t^{N-1} \notin \mathbb{K}[[\varphi(t),\psi(t)]]$. All power series of order $\geq N$ lie in $\mathbb{K}[[\varphi(t),\psi(t)]]$ which implies that any integer $\geq N$ is in $G(f)$. Thus $c \leq N$. To check that $c = N$ suppose to the contrary that $c < N$. Then $c+1 \leq N$ and $N-1 \in G(f)$ i.e. $N-1 = \operatorname{ord} g(\varphi(t),\psi(t))$ for a power series $g(\varphi(t),\psi(t)) \neq 0$. Multiplying $g(\varphi(t),\psi(t))$ by a constant we may assume that $g(\varphi(t),\psi(t)) = t^{N-1} + \gamma(t)$ where $\operatorname{ord}\gamma(t) \geq N$. Therefore $t^{N-1} = g(\varphi(t),\psi(t)) - \gamma(t) \in \mathbb{K}[[\varphi(t),\psi(t)]]$ since $\gamma(t) \in \mathbb{K}[[\varphi(t),\psi(t)]]$. A contradiction ■

Lemma 5.3. $\dim_{\mathbb{K}} \mathcal{O}_f/\mathcal{C} = \#(G(f) \cap [0, c-1])$.

Proof. By Lemma 5.2 we have $\mathcal{C} = \{\gamma = \gamma(t) \in \mathbb{K}[[\varphi(t),\psi(t)]] : \operatorname{ord}\gamma(t) \geq c\}$. Let $h = \#(G(f) \cap [0, c-1])$ and let $\varepsilon_1, \ldots, \varepsilon_h \in \mathbb{K}[[\varphi(t),\psi(t)]]$ be power series such that $\{\operatorname{ord}\varepsilon_1, \ldots, \operatorname{ord}\varepsilon_h\} = G(f) \cap [0, c-1]$. First, observe that $\varepsilon_1, \ldots, \varepsilon_h$ are linearly independent $\operatorname{mod} \mathcal{C}$: if $a_i \in \mathbb{K}$ are not all equal to zero then $\operatorname{ord}(\sum a_i\varepsilon_i) = \operatorname{ord}\varepsilon_{i_0} < c$ since $\operatorname{ord}\varepsilon_i \neq \operatorname{ord}\varepsilon_j$ for $i \neq j$. Therefore $\sum a_i\varepsilon_i \notin \mathcal{C}$. To check that $\varepsilon_1, \ldots, \varepsilon_h$ generate $\mathcal{O} \operatorname{mod} \mathcal{C}$ consider a power series $\gamma \not\equiv 0 \,(\operatorname{mod}\mathcal{C})$. Since $\gamma \notin \mathcal{C}$ we have $\operatorname{ord}\gamma < c$ and there is a unique $k \in \{1, \ldots, h\}$ such that $\operatorname{ord}\gamma = \operatorname{ord}\varepsilon_k$. Let $a_k \in \mathbb{K}$ be such that $\operatorname{ord}(\gamma - a_k\varepsilon_k) > \operatorname{ord}\gamma$. If $\operatorname{ord}(\gamma - a_k\varepsilon_k) \geq c$ then $\gamma \equiv a_k\varepsilon_k \,(\operatorname{mod}\mathcal{C})$, if not there is a $l \neq k$ such that $\operatorname{ord}(\gamma - a_k\varepsilon_k) = \operatorname{ord}\varepsilon_l$ and consequently $\operatorname{ord}(\gamma - a_k\varepsilon_k - a_l\varepsilon_l) > \operatorname{ord}(\gamma - a_k\varepsilon_k)$. Proceeding in this way we find constants $a_k, a_l, \ldots, a_p \in \mathbb{K} \setminus \{0\}$ such that $\operatorname{ord}(\gamma - a_k\varepsilon_k - a_l\varepsilon_l - \cdots - a_p\varepsilon_p) > c$. Therefore we get $\gamma = a_k\varepsilon_k + a_l\varepsilon_l + \cdots + a_p\varepsilon_p \;(\operatorname{mod}\mathcal{C})$ which ends the proof ■

Proof of Theorem 5.1

The semigroup $G(f)$ is symmetric by Theorem 4.5. Therefore by Lemmas 1.2 $\#(G(f) \cap [0, c-1]) = c/2$. Use Lemma 5.2 and 5.3 ■

Let us note the following corollary to Theorem 5.1.

Theorem 5.4. *Let* $\delta = \dim_{\mathbb{K}} \hat{\mathcal{O}}_f/\mathcal{O}_f$ (see [Hi]). *Then* $c = 2\delta$.

Proof. $\mathcal{C} \subset \mathcal{O}_f \subset \hat{\mathcal{O}}_f$ is a chain of vector spaces such that $\dim_{\mathbb{K}} \hat{\mathcal{O}}_f/\mathcal{C} < +\infty$. Therefore

$$\dim_{\mathbb{K}} \hat{\mathcal{O}}_f/\mathcal{C} = \dim_{\mathbb{K}} \mathcal{O}_f/\mathcal{C} + \dim_{\mathbb{K}} \hat{\mathcal{O}}_f/\mathcal{O}_f \,.$$

Use Theorem 5.1 ■

Remark 5.5. We define $c(f) = \dim_{\mathbb{K}} \hat{\mathcal{O}}_f/\mathcal{C}$ and $\delta(f) = \dim_{\mathbb{K}} \mathcal{O}_f/\mathcal{C}$ for any reduced curve $\{f = 0\}$. Let $f = f_1 \ldots f_r$ with irreducible, pairwise coprime $f_i \in \mathbb{K}[[x,y]]$ for $i = 1, \ldots, r$. Then

$$\begin{aligned} c(f) &= \sum_{i=1}^{r} c(f_i) + 2 \sum_{1 \leq i < j \leq r} i_0(f_i, f_j) \text{ and} \\ \delta(f) &= \sum_{i=1}^{r} \delta(f_i) + \sum_{1 \leq i < j \leq r} i_0(f_i, f_j) \,. \end{aligned}$$

Using Theorem 5.1 and the above formulas we check that $c(f) = 2\,\delta(f)$ for any reduced curve $\{f = 0\}$.

References

[A] Apéry, R., *Sur les branches superlinéaires des courbes algébraiques*, C. R. Acad. Sci., Paris 222, 1198–1200 (1946).

[Az] Azevedo, A., *The Jacobian ideal of a plane algebroid curve*, Thesis, Purdue University, Fort Wayne, Indiana 1967.

[An] Angermüller, G., *Die Wertehalbgruppe einer ebenen irreduziblen algebroiden Kurve*, Math. Z. 153 (1977), 267–282.

[F-Z] Flenner, H. and Zaidenberg, M., *On a class of rational cuspidal plane curves*, manuscripta math. 89, 439–459 (1996).

[G] Gorenstein, D., *An arithmetic theory of adjoint plane curves*, Trans. Amer. Math. Soc. 118 (1965), 454–471.

[He] Hefez, A., *Irreducible plane curve singularities, Real and complex singularities*, 1–120, Lecture Notes in Pure and App. Math., 232, Deckker, New York (2003).

[Hi] Hironaka, H., *On the arithmetic genera and effective genera of algebraic curves*, Mem. Coll. Sci. Univ. Kyoto. Se. A. Math. 30 (1957), 177–195.

[Pł] Płoski, A., *Introduction to the local theory of plane algebraic curves*, in Analytic and algebraic geometry (eds T. Krasiński and St. Spodzieja) Łódź University Press, Łódź 2013.

NIEROZKŁADALNE PŁASKIE KRZYWE ALGEBROIDALNE WEDŁUG R. APÉRY'EGO

Streszczenie. Celem tego opracowania jest przedstawienie metody Apéry'ego w lokalnej teorii krzywych algebraicznych. Dowodzimy, że dwie lokalne krzywe algebroidalne, nierozkładalne mają taką samą półgrupę dokładnie wtedy, gdy mają taki sam ciąg krotności. Ponadto dowodzimy za Azevedo, że lokalny pierścień nierozkładalnej krzywej płaskiej jest pierścieniem Gorensteina.

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