Waldschmidt constants

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XXXVIII Conference and Workshop on analytic and algebraic geometry
Łódź, January 10, 2017

Schneider-Lang Theorem in one variable

Theorem

Let $f_1, \dots, f_k$ be meromorphic functions in $\mathbb{C}$ with f_1, f_2 algebraically independent. Let $\mathbb{K}$ be a number field. Assume that for all $j = 1, \dots, k$

$$f_j' \in \mathbb{K}[f_1, \dots, f_k].$$

Then the set

$$S = \{z \in \mathbb{C} \ : z \text{ is not a pole of } f_j, f_j(z) \in \mathbb{K}, j = 1, \dots, k\}$$

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Corollary (Hermite-Lindemann)

For $\omega \in \mathbb{C}^*$ at least one of the numbers ω, $\exp(\omega)$ is transcendental.

Schwarz Lemma in one variable

Theorem

Let f be an analytic function in a disc $\{|z| \leq R\} \subset \mathbb{C}$ with at least N zeroes in a disc $\{|z| \leq r\}$ with $r < R$. Then

$$|f|_r \leq \left(\frac{3r}{R}\right)^N |f|_R,$$

where

$$|f|_\gamma = \sup_{|z|\leq\gamma} |f(z)|.$$

Schneider-Lang Theorem in several variables

Theorem (Bombieri 1970)

Let $f_1, \ldots, f_k$ be meromorphic functions in $\mathbb{C}^n$ with $f_1, \ldots, f_{n+1}$ algebraically independent. Let $\mathbb{K}$ be a number field. Assume that for all $i = 1, \ldots, n$, $j = 1, \ldots, k$

$$\frac{\partial}{\partial z_i} f_j \in \mathbb{K}[f_1, \ldots, f_k].$$

Then the set

$$S = \{z \in \mathbb{C}^n \ : z \text{ is not a pole of } f_j, f_j(z) \in \mathbb{K}, j = 1, \ldots, k\}$$

is contained in an algebraic hypersurface.

Hörmander version of Schwarz lemma in several variables

Theorem

Let $S \subset \mathbb{C}^n$ be a finite set. Let m be a positive integer. There exists $M(m) > 0$ such that there exists $r > 0$ such that for $R > r$ and a function f analytic in the ball $\{|z| \leq R\} \subset \mathbb{C}^n$ vanishing with multiplicity $\geq m$ at each point of S

$$|f|_r \leq \left(\frac{c(n) \cdot r}{R}\right)^{M(m)} |f|_R,$$

where $c(n)$ is a constant depending only on n.

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Problem

Make the statement effective. In particular: what is the maximal value of $M(m)$?

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Theorem (Moreau)

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$$|f|_r \leq \left(\frac{\exp(n) \cdot r}{R}\right)^{\alpha(mS)} |f|_R,$$

where $\alpha(mS)$ is the initial degree of $I_S^{(m)}$.

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Remark

The constant $\alpha(mS)$ is optimal.

Symbolic powers

Definition

Let $\mathbb{K}$ be a field and let $R = \mathbb{K}[x_0, \ldots, x_n]$ be the ring of polynomials. For a homogeneous ideal $0 \neq I \subsetneq R$ its *m-th symbolic power* is

$$I^{(m)} = \bigcap_{P \in \mathrm{Ass}(I)} \left(I^m R_P \cap R \right).$$

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Theorem (Zariski-Nagata)

Let $X \subset \mathbb{P}^n(\mathbb{K})$ be a projective variety (in particular reduced). Then $I(X)^{(m)}$ is generated by all forms which vanish along X to order at least m.

Symbolic powers of ideals of points

Let $Z = \{P_1, \ldots, P_s\}$ *be a finite set of points in* $\mathbb{P}^n(\mathbb{K})$*. Then*

$$I(Z) = I(P_1) \cap \ldots \cap I(P_s)$$

and

$$I(Z)^{(m)} = I(P_1)^m \cap \ldots \cap I(P_s)^m$$

for all $m \geq 1$*.*

The initial degree and the Waldschmidt constant

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For a graded ideal I its *initial degree* $\alpha(I)$ is the least number t such that $I_t \neq 0$.

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The *Waldschmidt constant* of I is the real number

$$\widehat{\alpha}(I) = \inf_{m \geq 1} \frac{\alpha(I^{(m)})}{m}.$$

Waldschmidt constants are hard to compute

Conjecture (Nagata)

Let I be a saturated ideal of $s \geq 10$ very general points in $\mathbb{P}^2(\mathbb{C})$. Then

$$\alpha(I^{(m)}) > m\sqrt{s}$$

for all $m \geq 1$.

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Equivalently

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Chudnovsky and Demailly Conjectures

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Remark

The Chudnovsky Conjecture is the $m = 1$ case of the Demailly Conjecture.

The containment problem

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Compare ordinary and symbolic powers of homogeneous ideals.

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More precisely, given I *determine all pairs* (m, r) *such that*
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Compare ordinary and symbolic powers of homogeneous ideals. More precisely, given I determine all pairs (m, r) such that
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$$I^r \subset I^{(m)} \quad \Leftrightarrow \quad r \geq m.$$

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Proposition

$$I^r \subset I^{(m)} \quad \Leftrightarrow \quad r \geq m.$$

Theorem (Ein-Lazarsfeld-Smith, Hochster-Huneke)

$$\textit{If } m \geq \operatorname{bight}(I) r, \; \textit{then } I^{(m)} \subset I^r.$$

Containment and Chudnovsky Conjecture

Theorem (Ein, Lazarsfeld, Smith; Hochster, Huneke)

Let I be a saturated ideal in $\mathbb{K}[x_0, \ldots, x_n]$. Then for all $m \geq nr$

$$I^{(m)} \subset I^r.$$

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Corollary

Let I be a saturated ideal of points in $\mathbb{P}^n(\mathbb{K})$. Then

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Theorem (Demailly)

In the set up as above

$$\widehat{\alpha}(I) \geq \frac{\alpha(I)(\alpha(I)+1) \cdot \ldots \cdot (\alpha(I)+n-1)}{n!\ \alpha(I)^{n-1}}.$$

Star configurations

Definition (Star configuration of points)

We say that $Z \subset \mathbb{P}^N$ is a *star configuration* of degree d (or a d-star for short) if Z consists of **all** intersection points of $d \geq N$ **general** hyperplanes in $\mathbb{P}^N$. By intersection points we mean the points which belong to exactly N of given d hyperplanes.

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Example (Bocci, Harbourne)

For points in the star configuration in $\mathbb{P}^n$, there is the equality in the Chudnovsky Conjecture.

Squarefree monomial ideals

Theorem (Bocci, Cooper, Guardo, Harbourne, Janssen, Nagel, Seceleanu, Van Tuyl, Vu 2015)

Let I be a squarefree monomial ideal with $\mathrm{bight}(I) = e$. Then

$$\widehat{\alpha}(I) \geq \frac{\alpha(I) + e - 1}{e}.$$

More evidence for the Chudnovsky Conjecture

Theorem (Esnault – Viehweg 1983)

Let I be a radical ideal of a finite set of points in $\mathbb{P}^n$ with $n \geq 2$. Let $k \leq m$ be two integers. Then

$$\frac{\alpha(I^{(k)}) + 1}{k + n - 1} \leq \frac{\alpha(I^{(m)})}{m},$$

in particular

$$\frac{\alpha(I^{(k)}) + 1}{k + n - 1} \leq \widehat{\alpha}(I).$$

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Corollary

In particular

$$\frac{\alpha(I)+1}{n} \leq \widehat{\alpha}(I),$$

so the Chudnovsky Conjecture holds in $\mathbb{P}^2$.

General points

Theorem (Dumnicki-Tutaj-Gasińska, Fouli-Manteo-Xie 2016)

The Chudnovsky Conjecture holds for general points in $\mathbb{P}^n$.

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Theorem (Malara, Szemberg, Szpond 2017)

The Demilly Conjecture holds for general points in $\mathbb{P}^n$.