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NUMERICAL SEMIGROUPS

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This paper is intended to give proofs of the results stated without proof in [GB-P2015]. We do this in Sections 1 and 2 following the approach due to Angermüller [An] (see also [GB-P2012]). In Section 3 we consider characteristic sequences satisfying the Abhyankar-Moh inequality which appears when studying the plane curves with one branch at infinity (see [A-M], [B-GB-P]).

In Section 4 we present some applications of numerical semigroups to plane curve singularities. In all the paper we denote by $\mathbb{N}$ the set of non-negative integers. If $a_0, \ldots, a_m \in \mathbb{N}$ then $\mathbb{N}a_0 + \cdots + \mathbb{N}a_m$ stands for the set of all integers of the form $q_0a_0 + \cdots + q_ma_m$, where $q_0, \ldots, q_m \in \mathbb{N}$. If $S \subset \mathbb{N}$, $S \neq \{0\}$ then $\gcd(S)$ denotes the greatest common divisor of all integers belonging to S.

1. Nice sequences

Let us begin with the following lemma.

Lemma 1.1. *Let $(v_0, \ldots, v_h)$ be a sequence of positive integers. Set $d_k = \gcd(v_0, \ldots, v_k)$ for $k = 0, \ldots, h$ and $n_k = d_{k-1}/d_k$ for $k = 1, \ldots, h$. Then for every $a \in \mathbb{Z}d_h$ we have Bézout's relation*

$$a = a_0v_0 + a_1v_1 + \cdots + a_hv_h$$

where $a_0, a_1, \ldots, a_h \in \mathbb{Z}$ and $0 \leqslant a_k < n_k$ for $k = 1, \ldots, h$. The sequence $(a_0, \ldots, a_h)$ is unique.

Proof. Existence. If $h = 0$ the lemma is obvious. Suppose that $h > 0$ and that the lemma is true for $h-1$. Since $d_h = \gcd(d_{h-1}, v_h)$ we can write for every $a \in (d_h)\mathbb{Z}$: $a = a'd_{h-1} + a''v_h$ with $a', a'' \in \mathbb{Z}$. For any integer l we have $a = (a' - lv_h)d_{h-1} + (a'' + ld_{h-1})v_h$. Thus we can take $a'' \geqslant 0$. Dividing a'' by $n_h = d_{h-1}/d_h$ we get $a'' = n_h a''' + a_h$ with $0 \leqslant a_h < n_h$. Therefore $a = a'd_{h-1} + (n_h a''' + a_h)v_h = (a' + \frac{v_h}{d_h}a''')d_{h-1} + a_h v_h$. By induction hypothesis we get $(a' + \frac{v_h}{d_h}a''')d_{h-1} = a_0v_0 + \cdots + a_{h-1}v_{h-1}$ with $0 \leqslant a_k < n_k$ for $0 \leqslant k \leqslant h-1$ and we are done.

Uniqueness. Suppose that $a_0v_0 + \cdots + a_hv_h = a'_0v_0 + \cdots + a'_hv_h$ with $0 \leqslant a_k, a'_k < n_k$ for $k > 0$. Let $a_h \leqslant a'_h$. Then $(a'_h - a_h)v_h \equiv 0 \bmod (v_0, \ldots, v_{h-1})\mathbb{Z}$ and $(a'_h - a_h)v_h \equiv 0 \pmod{d_{h-1}}$ which implies $(a'_h - a_h)\frac{v_h}{d_h} \equiv 0 \pmod{n_h}$. Therefore $a'_h - a_h \equiv 0 \pmod{n_h}$ and $a'_h - a_h = 0$ since $0 \leqslant a'_h - a_h < n_h$. Uniqueness follows by induction ∎

In what follows we assume that $d_h = \gcd(v_0, \ldots, v_h) = 1.$

We set

$$c = \sum_{k=1}^{h}(n_k - 1)v_k - v_0 + 1$$

and call c the virtual conductor of the sequence $(v_0, \ldots, v_h)$.

Property 1.2. *Let c be the virtual conductor of the sequence $(v_0, \ldots, v_h)$. Then $c \geqslant 0$ and $c = 0$ if and only if $v_k = d_k$ for all $k = 1, \ldots, h$ such that $n_k > 1$.*

Proof. Obviously we have $v_k \geqslant d_k$ for $k = 1, \ldots, h$. Therefore we get

$$c = \sum_{k=1}^{h}(n_k - 1)v_k - v_0 + 1 \geqslant \sum_{k=1}^{h}(n_k - 1)d_k - d_0 + 1 = 0\ .$$

Clearly $c = 0$ if and only if $v_k = d_k$ for all k such that $n_k > 1$ ∎

Property 1.3. (Brauer) *With the notation introduced above, if a is an integer, $a \geqslant c$ then $a \in \mathbb{N}v_0 + \cdots + \mathbb{N}v_h$.*

Proof. Let's write Bézouts relation for the integer a: $a = a_0v_0 + a_1v_1 + \cdots + a_hv_h$ where $0 \leqslant a_k \leqslant n_k - 1$ for $k = 1, \ldots, h$. Then $a_0v_0 = a - \sum_{k=1}^{h} a_kv_k \geqslant c - \sum_{k=1}^{h} a_kv_k = -v_0 + 1 + \sum_{k=1}^{h}(n_k - 1 - a_k)v_k \geqslant -v_0 + 1$. Consequently, we get $a_0 \geqslant \frac{-v_0+1}{v_0} = -1 + \frac{1}{v_0} > -1$ which implies $a_0 \geqslant 0$ ∎

Property 1.4. *Suppose that $lv_k \in \mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1}$ for an integer $l \geqslant 0$. Then $l \equiv 0 \pmod{n_k}$.*

Proof. If $lv_k \in \mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1}$ then $lv_k \equiv 0 \pmod{d_{k-1}}$ and $l(\frac{v_k}{d_k}) \equiv 0 \pmod{n_k}$. Since $\frac{v_k}{d_k}$ and n_k are coprime we get $l \equiv 0 \pmod{n_k}$ ∎

Definition 1.5. A sequence $(v_0, \ldots, v_h)$ is nice if $n_kv_k \in \mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1}$ for $k = 1, \ldots, h$.

Note that $n_1v_1 = (\frac{v_1}{d_1})v_0 \in \mathbb{N}v_0$. Hence the sequence (v_0, v_1) is nice. The sequence $(6, 7, 8)$ is not nice but the sequence $(6, 9, 7)$ is.

Property 1.6. *Let* $(v_0,\dots,v_h)$ *be a nice sequence. Then for every* $k\in\{1,\dots,h\}$*:* $v_k\notin \mathbb{N}v_0+\cdots+\mathbb{N}v_{k-1}$ *if and only if* $n_k>1$.

Proof. If $v_k\notin \mathbb{N}v_0+\cdots+\mathbb{N}v_{k-1}$ then $n_k>1$ by the definition of nice sequence. If $n_k>1$ then $v_k\notin \mathbb{N}v_0+\cdots+\mathbb{N}v_{k-1}$ by Property 1.4 ■

Proposition 1.7. *Let* $(v_0,\dots,v_h)$ *be a nice sequence,* c *the virtual conductor of* $(v_0,\dots,v_h)$*. Set* $G=\mathbb{N}v_0+\cdots+\mathbb{N}v_h$*. Then*

(i) *if* $a\in\mathbb{N}v_0+\cdots+\mathbb{N}v_k$ *then* $a=a_0v_0+\cdots+a_kv_k$ *with* $0\leqslant a_0$ *and* $0\leqslant a_i<n_i$ *for* $i=1,\dots,k$.

(ii) *For every* $a,b\in\mathbb{Z}$*: if* $a+b=c-1$ *then exactly one element of the pair* (a,b) *belongs to* G.

(iii) *The virtual conductor* c *equals the conductor of* G *i.e. all integers bigger than or equal to* c *are in* G *and* $c-1\notin G$.

(iv) c *is an even number and* $\#(\mathbb{N}\setminus G)=c/2$.

Proof. (i) If $k=0$ the assertion is obvious. Suppose that $k>0$ and that the property is true for $k-1$. By assumption we have $a=q_0v_0+\cdots+q_kv_k$ with $q_i\geqslant 0$ for $i=0,\dots,k$. By the Euclidean division of q_k by n_k we get $q_k=q_k'n_k+a_k$ with $0\leqslant a_k<n_k$. Thus $a=q_0v_0+\cdots+q_{k-1}v_{k-1}+q_k'n_kv_k+a_kv_k=a'+a_kv_k$ where $0\leqslant a_k<n_k$ and $a'\in\mathbb{N}v_0+\cdots+\mathbb{N}v_{k-1}$ since $n_kv_k\in\mathbb{N}v_0+\cdots+\mathbb{N}v_{k-1}$. Use the induction hypothesis.

(ii) Take two integers $a,b\in\mathbb{Z}$ such that $a+b=c-1$. Let us write Bézout's relation $a=a_0v_0+a_1v_1+\cdots+a_hv_h$ where $a_0\in\mathbb{Z}$ and $0\leqslant a_i<n_i$ for $i=1,\dots,k$. Then by the definition of c we get $b=c-1-a=-v_0+\sum_{k=1}^{h}(n_k-1)v_k-a_0v_0-\sum_{k=1}^{h}a_kv_k=-(a_0+1)v_0+\sum_{k=1}^{h}(n_k-1-a_k)v_k$. This is a Bézout's relation. To finish the proof it suffices to remark that exactly one element of the pair $(a_0,-a_0-1)$ is greater than or equal to zero.

(iii) By Property 1.3 all integers $\geqslant c$ are in G. Since $(c-1)+0=c-1$ and $0\in G$ we have $c-1\notin G$ by (ii).

(iv) The mapping $[0,c-1]\cap G\ni a\mapsto c-1-a\in[0,c-1]\cap(\mathbb{N}\setminus G)$ is bijective. Therefore we have $2\cdot\#([0,c-1]\cap G)=c$ and (iv) follows ■

Proposition 1.8. *Let* $(v_0,\dots,v_h)$ *be a sequence of positive integers such that* $n_kv_k\leqslant v_{k+1}$ *for* $k=1,\dots,h-1$*. Then* $(v_0,\dots,v_k)$ *is a nice sequence.*

Proof. Fix $k\in\{1,\dots,h-1\}$. Since $n_kv_k=d_{k-1}\frac{v_k}{d_k}\equiv 0\,(\mathrm{mod}\,d_{k-1})$ by Lemma 1.1 we can write Bézout's identity

$$n_kv_k=a_0v_0+a_1v_1+\cdots+a_{k-1}v_{k-1}$$

where $a_0\in\mathbb{Z}$ and $0\leqslant a_i<n_i$ for $i=1,\dots,k-1$. Therefore, we get $a_0v_0=n_kv_k-a_1v_1-\cdots-a_{k-1}v_{k-1}\geqslant n_kv_k-(n_1-1)v_1-\cdots-(n_{k-1}-1)v_{k-1}=n_kv_k-[(n_1v_1-v_1)+\cdots+(n_{k-1}v_{k-1}-v_{k-1})]>n_kv_k-[(v_2-v_1)+\cdots+(v_k-v_{k-1})]=n_kv_k-v_k+v_1>0$ which implies $a_0>0$ ■

Remark 1.9. In fact we have proved the following property, stronger than "$(v_0,\dots,v_h)$ is nice": if $n_kv_k=a_0v_0+a_1v_1+\cdots+a_{k-1}v_{k-1}$ is a Bézout's relation then $a_0>0$.

2. Semigroups of natural numbers

A subset G of $\mathbb{N}$ closed under addition and containing 0 is called a *semigroup*. In what follows we assume $G \neq \{0\}$. A semigroup is numerical if $\gcd(G) = 1$.

Lemma 2.1. *Let G be a semigroup and let $v_0 \in G$, $v_0 > 0$. If $G \neq \mathbb{N}v_0$ then there exists a unique sequence $v_1, \ldots, v_h$ such that*

(i) $G \neq \mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1}$ *and* $v_k = \min(G \setminus (\mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1}))$ *for* $k = 1, \ldots, h$

(ii) $G = \mathbb{N}v_0 + \cdots + \mathbb{N}v_h$.

Proof. Observe that if $v_1, \ldots, v_k$ satisfy conditions (i) then $v_k \not\equiv v_l \pmod{v_0}$ for $l < k$. Indeed, from $v_k \equiv v_l \pmod{v_0}$ we get $v_k = v_l + qv_0$ with $q \in \mathbb{N}$ which implies $v_k \in \mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1}$ since $l < k$. Therefore the conditions (i) define a finite sequence. It suffices to take as $(v_1, \ldots, v_h)$ the longest sequence with property (i) ∎

We call the sequence $(v_0, v_1, \ldots, v_h)$ the v_0-*minimal system of generators of* G (if $G = \mathbb{N}v_0$ then the v_0-minimal system of generators is (v_0)). If $v_0 = \min(G \setminus \{0\})$ then we say that $(v_0, v_1, \ldots, v_h)$ is the *minimal sequence of generators of* G. Clearly $\gcd G = \gcd(v_0, v_1, \ldots, v_h) = d_h$ ($d_h = 1$ if G is a numerical semigroup).

Lemma 2.2. *Let $(v_0, \ldots, v_h)$ be a v_0-minimal system of generators of the semigroup G. Then*

(i) $v_1 < \cdots < v_h$,

(ii) $\min(G \setminus \{0\}) = \min(v_0, v_1)$,

(iii) *if $v \in G$ and $v < v_k$ for a $k > 0$ then $v \in \mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1}$,*

(iv) *each v_k, $k > 0$ is an irreducible element of G, that is v_k is not a sum of two nonzero elements of the semigroup G.*

Proof. (i) We have $G \setminus (\mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-2}) \supset G \setminus (\mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1})$ for $k \geqslant 2$. Since $v_k \in G \setminus (\mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1})$ by the definition of v_k, we have $v_k \in G \setminus (\mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-2})$ and $v_k \geqslant \min(G \setminus (\mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-2})) = v_{k-1}$ for $k \geqslant 2$. Thus we get $v_{k-1} < v_k$ since $v_{k-1} \neq v_k$.

(ii) $\min(G \setminus \{0\}) = \min(v_0, v_1, \ldots, v_h) = \min(v_0, v_1)$ by (i).

(iii) If $v \in G \setminus (\mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1})$ then $v = q_0v_0 + \cdots + q_kv_k + \cdots + q_hv_h$ with $q_l \neq 0$ for an index $l \geqslant k$. Thus $v \geqslant q_lv_l \geqslant v_l \geqslant v_k$ which proves (iii).

(iv) Suppose that $v_k = v' + v''$ with nonzero $v', v'' \in G$. Therefore $v', v'' < v_k$ and $v_k = v' + v'' \in \mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1}$ by (iii). This contradicts the definition of v_k ∎

Lemma 2.3. *Let $(v_0, \ldots, v_h)$ be a sequence of positive integers such that $v_1 < \cdots < v_h$ and $v_k \notin \mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1}$ for $k = 1, \ldots, h$. Then $(v_0, \ldots, v_h)$ is a v_0-minimal system of generators of the semigroup $G = \mathbb{N}v_0 + \cdots + \mathbb{N}v_h$.*

Proof. We check like in the proof of Lemma 2.2 (iii) that $v_k = \min(G \setminus (\mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1}))$ for $k = 1, \ldots, h$ ∎

Proposition 2.4. *Let G be a numerical semigroup with v_0-minimal system of generators $(v_0, \ldots, v_h)$. Suppose that $n_kv_k \leqslant v_{k+1}$ for $k = 1, \ldots, k-1$. Then*

(i) $n_k > 1$ *for* $k = 1, \ldots, h$,

(ii) $n_k v_k < v_{k+1}$ *for* $k = 1, \dots, h-1$,

(iii) *the minimal system of generators of* G *is* $(v_0, v_1, \dots, v_h)$ *if* $v_0 < v_1$, $(v_1, v_0, \dots, v_h)$ *if* $v_1 < v_0$ *and* $v_0 \not\equiv 0 \,(\mathrm{mod}\, v_1)$, $(v_1, \dots, v_h)$ *if* $v_1 < v_0$ *and* $v_0 \equiv 0 \,(\mathrm{mod}\, v_1)$.

Proof. By Proposition 1.8 the sequence $(v_0, v_1, \dots, v_h)$ is nice. Since $v_{k+1} \notin \mathbb{N}v_0 + \cdots + \mathbb{N}v_k$ we have $n_k > 1$. Property (ii) is obvious. To check (iii) observe that the inequality $n_1 v_1 < v_2$ implies $v_0 < v_2$ since $n_1 v_1 = v_0(\frac{v_1}{d_1})$ and recall that by Lemma 2.2 (ii) we have $\min(G \setminus \{0\}) = \min(v_0, v_1)$ ■

3. Characteristic sequences

A sequence of positive integers $(r_0, \dots, r_h)$ is said to be a *characteristic sequence* if it satisfies the following two axioms:

1. Set $d_k = \gcd(r_0, \dots, r_k)$ for $0 \leqslant k \leqslant h$. Then $d_k > d_{k+1}$ for $0 \leqslant k < h$ and $d_h = 1$.
2. Set $n_k = d_{k-1}/d_k$ for $k = 1, \dots, h$. Then $n_k r_k < r_{k+1}$ for $1 \leqslant k < h$.

We call r_0 the initial term of the characteristic sequence $(r_0, \dots, r_h)$. Let $G = \mathbb{N}r_0 + \cdots + \mathbb{N}r_h$ be the semigroup generated by a characteristic sequence. Then $(r_0, \dots, r_h)$ is a r_0-minimal system of generators of G (cf. Lemma 2.3). In particular G and r_0 determine the sequence $(r_0, \dots, r_h)$.

Proposition 3.1. *Let* G *be the semigroup generated by a characteristic sequence* $(r_0, \dots, r_h)$. *Then the conductor* c *of* G *equals*

$$c = \sum_{k=1}^{h} (n_k - 1) r_k - r_0 + 1 \,.$$

The semigroup G *is symmetric: if* $a, b \in \mathbb{Z}$ *and* $a + b = c - 1$ *then exactly one element of the pair* (a, b) *belongs to* G.

Proof. The proposition follows from Propositions 1.7 and 1.8.

A characteristic sequence $(r_0, \dots, r_h)$ has the Abhyankar-Moh property (in short: the AM property) if it satisfies the inequality

$$d_{h-1} r_h < r_0^2 \,.$$

For every such sequence we define the associated sequence $(\delta_0, \dots, \delta_h)$ by putting

$$\delta_0 = r_0, \quad \delta_k = \frac{r_0^2}{d_{k-1}} - r_k \text{ for } 1 \leqslant k \leqslant h \,.$$

Lemma 3.2. *The associated sequence* $(\delta_0, \dots, \delta_h)$ *satisfies the following properties*

1) $\delta_k > 0$ *and* $\gcd(\delta_0, \dots, \delta_k) = d_k$ *for* $0 \leqslant k \leqslant h$,

2) $n_k \delta_k > \delta_{k+1}$ *for* $1 \leqslant k < h$.

3) *If* γ *is the virtual conductor of the sequence* $(\delta_0, \dots, \delta_h)$ *then* $\gamma = (r_0 - 1)(r_0 - 2) - c$.

Proof. We have $\delta_k = \dfrac{r_0^2 - d_{k-1}r_k}{d_{k-1}} \geqslant \dfrac{r_0^2 - d_{h-1}r_h}{d_{k-1}} > 0$. The second part of property 1) follows by induction on k. Since $\gcd(d_{k-1}, \delta_k) = d_k$ we get

$$\gcd(\delta_0, \ldots, \delta_k) = \gcd(\gcd(\delta_0, \ldots, \delta_{k-1}), \delta_k) = \gcd(d_{k-1}, \delta_k) = d_k\ .$$

To check 2) it suffices to observe that the inequalities $n_k\delta_k > \delta_{k+1}$ and $n_k r_k < r_{k+1}$ are equivalent. Recall that $\gamma = \sum_{k=1}^{h}(n_k - 1)\delta_k - \delta_0 + 1$. Thus we get

$$\gamma = \sum_{k=1}^{h}(n_k - 1)\left(\frac{r_0^2}{d_{k-1}} - r_k\right) - r_0 + 1 = \sum_{k=1}^{h}(n_k - 1)\frac{r_0^2}{d_{k-1}} - r_0 + 1 - \sum_{k=1}^{h}(n_k - 1)r_k$$

$$= (r_0 - 1)^2 - \sum_{k=1}^{h}(n_k - 1)r_k = (r_0 - 1)(r_0 - 2) - c \ \blacksquare$$

Proposition 3.3. *Suppose that $(r_0, \ldots, r_h)$ is a characteristic sequence with the AM property. Let c be the conductor of the semigroup $\mathbb{N}r_0 + \cdots + \mathbb{N}r_h$. Then $c \leqslant (r_0 - 1)(r_0 - 2)$ with equality if and only if $r_k = \frac{r_0^2}{d_{k-1}} - d_k$ for $1 \leqslant k \leqslant h$.*

Proof. By the third part of Lemma 3.2 $c = (r_0 - 1)(r_0 - 2) - \gamma \leqslant (r_0 - 1)(r_0 - 2)$ since $\gamma \geqslant 0$ by Property 1.2. The equality $c = (r_0 - 1)(r_0 - 2)$ is equivalent to $\gamma = 0$ which again by Property 1.2 is equivalent to $\delta_k = d_k$. Hence we get $r_k = \frac{r_0^2}{d_{k-1}} - d_k$ for $1 \leqslant k \leqslant h$ $\blacksquare$

Lemma 3.2 and Proposition 3.3 are due to [B-GB-P].

Remark 3.4. Although every characteristic sequence $(r_0, \ldots, r_h)$ is nice (see Proposition 1.8) the sequence $(\delta_0, \ldots, \delta_h)$ associated with an Abhyankar-Moh sequence is not nice, in general. The following example is due to J. Gwoździewicz: $(r_0, r_1, r_2) = (10, 4, 49)$ has the AM property but the associated sequence $(\delta_0, \delta_1, \delta_2) = (10, 6, 1)$ is not nice. See also the Barrolleta's example given in [B-GB-P].

4. Semigroups of plane branches

Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic and let $\mathbb{K}[[x, y]]$ be the ring of formal power series in two wariables x, y with coefficients in $\mathbb{K}$. For any $f, g \in \mathbb{K}[[x, y]]$ we define the intersection multiplicity $i_0(f, g)$ by putting

$$i_0(f, g) = \dim_{\mathbb{K}} \mathbb{K}[[x, y]] \big/ (f, g)\ .$$

If f and g are without constant term then $i_0(f, g) < +\infty$ if and only if f, g have no common factor h, $h(0, 0) = 0$. For any irreducible power series $f \in \mathbb{K}[[x, y]]$ we put

$$G(f) = \{i_0(f, g) : g \text{ runs over all power series such that } g \not\equiv 0 \,(\mathrm{mod}\, f)\}\ .$$

Clearly $G(f)$ is a semigroup. We call $G(f)$ the semigroup associated with the branch $f = 0$.

Theorem 4.1. (Bresinsky-Angermüller Semigroup Theorem)

1. *Let $f = f(x, y)$ be an irreducible power series. Suppose that $n = i_0(f, x) < +\infty$. Then the semigroup $G(f)$ of the branch $f = 0$ is generated by a characteristic sequence with the initial term n.*
2. *Let $G \subset \mathbb{N}$ be a semigroup generated by a characteristic sequence with the initial term $n > 0$. Then there exists an irreducible power series $f = f(x, y)$ such that $i_0(f, x) = n$ and $G(f) = G$.*

A characteristic-blind proof of the above theorem is given in [GB-P2012].

Two branches $f = 0$ and $g = 0$ are equisingular if and only if $G(f) = G(g)$. The Abhyankar-Moh inequality appears when studing the plane curves with one branch at infinity (see, for example [B-GB-P]). Here we present a characterization of the AM inequality in terms of pencils of plane local curves.

Theorem 4.2. *Let $f \in \mathbb{K}[[x, y]]$ be an irreducible power series, $n = i_0(f, x) < +\infty$ and let $G(f) = \mathbb{N}r_0 + \cdots + \mathbb{N}r_h$ where $(r_0, \ldots, r_h)$ is a characteristic sequence with the initial term $r_0 = n$. Suppose that* $\operatorname{char}\mathbb{K} = 0$. *Let $f_t = f - tX^n$. Then the following two conditions are equivalent:*

(AM) $d_{h-1}r_h < n^2$,

(E) *the pencil $(f_t : t \in \mathbb{K})$ is equisingular i.e. f_t are irreducible and $G(f_t) = G(f)$ for $t \in \mathbb{K}$.*

Proof. See [GB-P2004], Section 5, p. 124.

Let $F(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x) \in \mathbb{K}[x, y]$ be a polynomial of degree $n > 1$, irreducible in $\mathbb{K}[x, y]$. Assume, after possibly a change of variables, that $\deg a_k(x) < k$ for $k = 1, \ldots, n$. Hence $F_0(x_0, y_0) = y_0^n + x_0 a_1(\frac{1}{x_0})y_0^{n-1} + \cdots + x_0^n a_n(\frac{1}{x_0}) \in \mathbb{K}[x_0, y_0]$ is a distinguished polynomial. In what follows we assume that the polynomial $F(x, y)$ is *irreducible at infinity* i.e. $F_0(x_0, y_0)$ is irreducible in $\mathbb{K}[[x_0, y_0]]$. Given a polynomial $G(x, y) \in \mathbb{K}[x, y]$, we set

$$I(F, G) = \dim_{\mathbb{K}} \mathbb{K}[x, y] \big/ (F, G)$$

and call

$$\{I(F, G) : G \text{ runs over all polynomials such that } G \not\equiv 0 \,(\operatorname{mod} F)\}$$

the degree semigroup associated with the affine curve $F = 0$.

Theorem 4.3. (Abhyankar-Moh Degree Semigroup Theorem)
Suppose that $\mathbb{K}$ is an algebraically closed field of characteristic zero and keep the notation and assumption introduced above. Then the n-minimal system of generators of the semigroup $G(F_0)$ has the AM property, the associated sequence $(\delta_0, \ldots, \delta_h)$ is nice and generates the degree semigroup of the affine curve $F = 0$.

Proof. See [R].

For more information the reader is referred to [B-GB-P] and [RL].

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PÓŁGRUPY NUMERYCZNE

Półgrupą numeryczną nazywamy półgrupę liczb naturalnych $G \subset \mathbb{N}$ taką, że $\mathrm{NWP}(G) = 1$. Takie półgrupy występują w teorii osobliwości. W tym artykule opisujemy za Angermüllerem półgrupy stowarzyszone z osobliwościami krzywych płaskich a następnie opisujemy półgrupy, których ciągi generatorów spełniają nierówność Abhyankara-Moha, podstawową w teorii krzywych płaskich o jednej gałęzi w nieskończoności.

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