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THE MILNOR NUMBER OF A PLANE ALGEBROID CURVE

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Introduction

The aim of this note is to present an elementary approach to the local invariants of the plane curve singularities. We use only basic facts concerning formal power series such as the Weierstrass Preparation Theorem, Hensel's lemma and Puiseux's Theorem. We get the main properties of the Milnor number with the aid of Tessier's lemma (a particular case of a formula due to Teissier [9]). In proving the local version of the double–point divisor we follow van der Waerden's ideas [10].

1. Plane algebroid curves.

We review here some basic notions from the local theory of algebraic curves. For more details we refer the reader to [8].

Let $f \in \mathbb{C}[[x,y]]$ be a non–zero power series without constant term. An algebroid curve $f = 0$ is defined to be the ideal generated by f in $\mathbb{C}[[x,y]]$. We say that $f = 0$ is irreducible (reduced) if $f \in \mathbb{C}[[x,y]]$ is irreducible (f has no multiple factors). The irreducible curves are also called branches. If $f = f_1^{k_1} \dots f_s^{k_s}$ with non–associated irreducible factors f_i then we refer to $f_i = 0$ as the branches or components of $f = 0$. The order $\operatorname{ord} f$ of the power series f is, by definition, the multiplicity of the curve $f=0$. The initial form $\operatorname{in} f$ of f determines the tangent lines of $f=0$. If $f=0$ is irreducible then it has only one tangent line i. e. $\operatorname{in} f = l^{\operatorname{ord} f}$ where l is a linear form.

A local analytic transformation ϕ is given by

$$\phi(x,y) = (ax+by+\dots,\, a'x+b'y+\dots)$$

where $ab'-a'b \neq 0$ and the dots denote terms of order >1 in x, y. The map $f \to f\circ\phi$ is an isomorphism of the ring $\mathbb{C}[[x,y]]$.

Two curves $f=0$ and $g=0$ are said to be analytically equivalent if there is a local analytic transformation ϕ such that $f\circ\phi = g\cdot\text{unit}$. A function defined on the set of reduced curves is an analytic invariant if it is constant on analytically equivalent curves.

The multiplicity $\operatorname{ord} f$ of $f=0$ and the number $r(f)$ of the distinct branches of $f=0$ are analytic invariants.

Example. Any irreducible branch of multiplicity 2 is analytically equivalent to the curve $x^2+y^n=0$ where $n>1$ is an odd number.

For any power series $f, g \in \mathbb{C}[[x,y]]$ we define the intersection number $(f,g)_0$ by putting

$$(f,g)_0 = \dim_{\mathbb{C}} \mathbb{C}[[x,y]]/(f,g)$$

where (f,g) is the ideal of $\mathbb{C}[[x,y]]$ generated by f and g. If f, g are non–zero power series without constant term then $(f,g)_0<\infty$ if and only if the curves $f=0$ and $g=0$ are free from common branches. The following properties are basic

(i) if ϕ is a local analytic transformation then $(f,g)_0 = (f\circ\phi, g\circ\phi)_0$,

(ii) $(f,gh)_0 = (f,g)_0 + (f,h)_0$,

(iii) (“The Basic Inequality”) $(f,g)_0 \geq (\operatorname{ord} f)(\operatorname{ord} g)$. The equality $(f,g)_0 = (\operatorname{ord} f)(\operatorname{ord} g)$ holds if and only if the tangents to $f=0$ are all different from the tangents to $g=0$.

Let $f=f(x,y)\in\mathbb{C}[[x,y]]$ be an irreducible power series, y–distinguished with order $n\geq 1$ i. e. such that $\operatorname{ord} f(0,y)=n$. By Puiseux's Theorem there is a power series $y(t)\in\mathbb{C}[[t]]$ without constant term such that $f(t^n,y(t))=0$. We call $p(t)=(t^n,y(t))$ Puiseux's parametrization of the branch $f(x,y)=0$.

Proposition 1.1. *If $p(t)$ is Puiseux's parametrization of $f(x,y)=0$ then $(f,g)_0 = \operatorname{ord} g(p(t))$ for any $g\in\mathbb{C}[[x,y]]$.*

Proof. By the Weierstrass Preparation theorem we may assume that f is a distinguished polynomial. Let $n=1$. Then $f(x,y)=y-y(x)$, $y(0)=0$ and we have to check that $(f,g)_0=\operatorname{ord} g(x,y(x))$. For every $g\in\mathbb{C}[[x,y]]$ we put $\widetilde{g}(x,y)=g(x,y+y(x))$. The mapping $g\mapsto\widetilde{g}$ is an isomorphism of the ring $\mathbb{C}[[x,y]]$ and by Property (i) we get $(f,g)_0=(y,\widetilde{g}(x,y))_0=\dim_{\mathbb{C}}\mathbb{C}[[x,y]]/(y,\widetilde{g}) = \dim_{\mathbb{C}}\mathbb{C}[[x]]/\widetilde{g}(x,0) = \operatorname{ord}\widetilde{g}(x,0)=\operatorname{ord} g(x,y(x))$, hence the case $n=1$ follows.

Let $n > 1$. One checks directly that any power series $\phi(t,y) \in \mathbb{C}[[t,y]]$ can be uniquely represented in the form

$$\phi(t,y) = \phi_0(t^n,y) + \phi_1(t^n,y)t + \cdots + \phi_{n-1}(t^n,y)t^{n-1}$$

Let $F(t,y) = f(t^n,y)$ and $G(t,y) = g(t^n,y)$. Using the above remark we check that $(F,G)_0 = n(f,g)_0$. By Puiseux's Theorem $F(t,y) = \prod_{\epsilon^n=1}(y - y(\epsilon t))$, hence $n(f,g)_0 = (F,G)_0 = \sum_{\epsilon^n=1}(y - y(\epsilon t), G)_0 = \sum_{\epsilon^n=1} \operatorname{ord} G(t, y(\epsilon t)) = n \operatorname{ord} g(t^n, y(t))$ and we are done.

Note. In proving Proposition 1.1 we have used only (i) and (ii). The Basic Inequality can be checked easily with the aid of 1.1.

2. The Milnor number

For every power series $f \in \mathbb{C}[[x,y]]$ without constant term we define the Milnor number $\mu(f)$ by putting

$$\mu(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)_0$$

Note that $\mu(f) < \infty$ if and only if f has no multiple factors. We will get the main properties of the Milnor number from Teissier's lemma:

Lemma 2.1. *Let $f = f(x,y)$ be an y–distinguished power series with no multiple factors. Then*

$$\left(f, \frac{\partial f}{\partial y}\right)_0 = \mu(f) + (f,x)_0 - 1.$$

Proof. Let $\partial f/\partial y = g_1 \ldots g_s$ with irreducible $g_1, \ldots, g_s \in \mathbb{C}[[x,y]]$ and let $p_i(t) = (x_i(t), y_i(t))$ be Puiseux's parametrization of the branch $g_i = 0$. From $\partial f/\partial y(p_i(t)) = 0$ in $\mathbb{C}[[t]]$ we get $\frac{d}{dt} f(p_i(t)) = \frac{\partial f}{\partial x}(p_i(t))\frac{dx_i}{dt}$, hence $\operatorname{ord} f(p_i(t)) = \operatorname{ord} \partial f/\partial x(p_i(t)) + \operatorname{ord} x_i(t)$ for $i = 1, \ldots, s$. Using Proposition 2.1 we get

$$\begin{aligned}
\left(f, \frac{\partial f}{\partial y}\right)_0 &= \sum_{i=1}^{s} (f, g_i)_0 = \sum_{i=1}^{s} \operatorname{ord} f(p_i(t)) = \sum_{i=1}^{s} \operatorname{ord} \frac{\partial f}{\partial x}(p_i(t)) + \sum_{i=1}^{s} \operatorname{ord} x_i(t) \\
&= \sum_{i=1}^{s} \left(\frac{\partial f}{\partial x}, g_i\right)_0 + \sum_{i=1}^{s} (x, g_i)_0 = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)_0 + \left(x, \frac{\partial f}{\partial y}\right)_0 \\
&= \mu(f) + \operatorname{ord} f(0,y) - 1 = \mu(f) + (f,x)_0 - 1.
\end{aligned}$$

Property 2.2. *The Milnor number is an analytic invariant i. e. 1)* $\mu(f) = \mu(f \circ \phi)$ *if* ϕ *is a local analytic transformation, 2) if* $g = f \cdot unit$*, then* $\mu(f) = \mu(g)$.

Proof. The first part we leave to the reader. To check the second part assume that f is y–distinguished. We have $(f, \partial f/\partial y)_0 = (g, \partial g/\partial y)_0$ and $(f, x)_0 = (g, x)_0$ by properties of intersection numbers quoted in Section 1. Therefore 2) follows from Teissier's Lemma.

Property 2.3. *If* $f = f_1 \dots f_m$ *is a product of distinct* f_i*, then*

$$\mu(f) + m - 1 = \sum_{i=1}^{m} \mu(f_i) + 2 \sum_{1 \le i < j \le m} (f_i, f_j)_0$$

Proof. By basic properties of intersection numbers we get

$$\left(f, \frac{\partial f}{\partial y}\right)_0 = \left(f_1, \frac{\partial f_1}{\partial y}\right)_0 + \cdots + \left(f_m, \frac{\partial f_m}{\partial y}\right)_0 + 2 \sum_{1 \le i < j \le m} (f_i, f_j)_0.$$

To obtain the formula it suffices to apply Teissier's lemma to power series f, $f_1, \dots, f_m$.

Let $f = f(x,y) \in \mathbb{C}[[x,y]]$ be an irreducible power series of order $n > 1$. We may assume that $\operatorname{in} f = y^n$ i. e. $f(x,y) = y^n +$ terms of order greater than n. Let u be a variable. The power series $\widehat{f}(x,u) \in \mathbb{C}[[x,u]]$ is a proper transform of $f(x,y)$ (by the quadratic transformation $y = xu$, $x = x$) if $f(x, xu) = x^n \widehat{f}(x,u)$. Note that $\operatorname{ord} \widehat{f}(0,u) = \operatorname{ord} f(0,y) = n$.

Property 2.4. *If* f *is an irreducible power series,* $\widehat{f}$ *its proper transform, then* $\mu(f) = (\operatorname{ord} f)(\operatorname{ord} f - 1) + \mu(\widehat{f})$.

Proof. If $p(t) = (t^n, y(t))$ is Puiseux's parametrization of $f(x,y) = 0$ then $\operatorname{ord} y(t) > n$ and $\widehat{p}(t) = (t^n, y(t)/t^n)$ is Puiseux's parametrization of $\widehat{f}(x,u) = 0$. Using Proposition 2.1 we check that $(f, \partial f/\partial y)_0 = (\widehat{f}, \partial \widehat{f}/\partial y)_0 + n(n-1)$, hence $\mu(f) = n(n-1) + \mu(\widehat{f})$ by Teissier's Lemma.

By repeated application of Property 2.4 we get

Property 2.5. *For every irreducible power series* $f \in \mathbb{C}[[x,y]]$ *there exist a sequence of irreducible series* $f_0, \dots, f_s$ *such that* $f_0 = f$, $f_{i+1} = 0$ *is the proper transform of the branch* $f_i = 0$, $f_s = 0$ *is non-singular i. e.* $\operatorname{ord} f_s = 1$. *Moreover* $\mu(f) = \sum_{i=1}^{s} (\operatorname{ord} f_i)(\operatorname{ord} f_i - 1)$.

Note that the above formula for $\mu(f)$ implies that the Milnor number of a branch is always an even number. Now we may formulate the main result of this section

Theorem 2.6. *There is a unique analytic invariant δ defined on reduced algebroid curves such that*

(i) *if* $\operatorname{ord} f = 1$ *then* $\delta(f) = 0$,

(ii) *if* $f = 0$ *is an irreducible, singular curve then* $\delta(f) = (\operatorname{ord} f)(\operatorname{ord} f - 1)/2 + \delta(\widehat{f})$,

(iii) *if* $f = f_1 \dots f_r$ *is a product of distinct irreducible power series then* $\delta(f) = \sum_{i=1}^{r} \delta(f_i) + \sum_{1 \le i < j \le r} (f_i, f_j)_0$.

Moreover the Milnor formula $\mu(f) = 2\delta(f) - r(f) + 1$ *holds for any reduced curve* $f = 0$.

Proof. The uniqueness of δ follows immediately from Property 3.5 by induction on the number of local quadratic transformations needed to desingularize a branch. The existence of δ and the Milnor formula we prove simultaneously by putting $\delta(f) = (\mu(f) + r(f) - 1)/2$ and using the properties of the Milnor number proved above.

Note. For any reduced curve $f = 0$ we consider the local ring $\mathcal{O}_f = \mathbb{C}[[x, y]]/(f)$. Let $\mathcal{M}_f$ be the total quotient ring of $\mathcal{O}_f$ and let $\widehat{\mathcal{O}}_f$ be the integral closure of $\mathcal{O}_f$ in $\mathcal{M}_f$. Then we have Hironaka's formula: $\delta(f) = \dim_{\mathbb{C}} \widehat{\mathcal{O}}_f / \mathcal{O}_f$.

Indeed, according to Hironaka, the invariant $f \to \dim_{\mathbb{C}} \widehat{\mathcal{O}}_f / \mathcal{O}_f$ has properties (i), (ii) and (iii) listed in Theorem 2.6.

Example. The reduced curve $f = 0$ has an ordinary r–fold singularity if it has r branches, all non–singular and intersecting each other with multiplicity 1. For such a curve we have $\mu = (r-1)^2$ and $\delta = r(r-1)/2$.

3. Noether's Theorem

In this section we assume that $f \in \mathbb{C}[[x, y]]$ is a power series with no multiple factors. If $f = f_1 \dots f_r$ is a product of irreducible factors $f_i \in \mathbb{C}[[x, y]]$ then we set

$$c_i(f) = \mu(f_i) + \sum_{j \ne i} (f_i, f_j)_0 \quad \text{for } i = 1, \dots, r$$

A curve $\Psi = 0$ is said to be an adjoint to $f = 0$ if $(f_i, \Psi)_0 \ge c_i(f)$ for $i = 1, \dots, r$.

Example. Let $f = 0$ be an ordinary r–fold singularity. Then $\Psi = 0$ is an adjoint to $f = 0$ if and only if $\operatorname{ord} \Psi \ge r - 1$.

The following result is known as Noether's Theorem on the double-point divisor. Let g, $h \in \mathbb{C}[[x, y]]$.

Theorem 3.1. *Suppose that the curves* $f = 0$ *and* $g = 0$ *are without common component. If* h *satisfies Noether's conditions:*

$$(f_i, h)_0 \ge (f_i, g)_0 + c_i(f) \quad \textit{for } i = 1, \dots, r$$

then h *belongs to the ideal* (f, g) *generated by* f, g *in the ring* $\mathbb{C}[[x, y]]$.

Let us write $h = \phi f + \psi g$ with ϕ, $\psi \in \mathbb{C}[[x,y]]$. Then Noether's conditions imply that $\psi = 0$ is an adjoint to $f = 0$. In connection with Noether's Theorem let us note

Theorem 3.2. *Let $f \in \mathbb{C}[[x,y]]$ be an irreducible power series. Then there does not exist $\Psi \in \mathbb{C}[[x,y]]$ such that $(f,\Psi)_0 = \mu(f) - 1$. Let $h \in \mathbb{C}[[x,y]]$ be such that $(f,h)_0 = (f,g)_0 + \mu(f) - 1$, then $h \notin (f,g)\mathbb{C}[[x,y]]$.*

The second part of (3.2) follows easily from the first. Indeed, if we had $h = \phi f + \psi g$ with ϕ, $\psi \in \mathbb{C}[[x,y]]$ and $(f,h)_0 = (f,g)_0 + \mu(f) - 1$ then we would get $(f,\psi)_0 = \mu(f) - 1$, contradiction with the first part of 3.2.

Let us pass now to the proofs of (3.1) and (3.2). Let $F(u,y)$, $G(u,y)$, $H(u,y) \in \mathbb{C}[[u]][y]$ where u is a variable. Assume that $F(u,y) = \prod_{i=1}^{n}(y - y_i(u))$ in $\mathbb{C}[[u]][y]$ and $y_i(u) \neq y_j(u)$ for $i \neq j$.

Lemma 3.3. *If $\operatorname{ord} H(u,y_i(u)) \geq \operatorname{ord} \frac{\partial F}{\partial y}(u,y_i(u))G(u,y_i(u))$ for $i = 1,\dots,n$, then $H(u,y) \in (F(u,y),G(u,y))\mathbb{C}[[u]][y]$.*

Proof. Let

$$\Psi(u,y) = \sum_{i=1}^{n} \frac{H(u,y_i(u))}{\frac{\partial F}{\partial y}(u,y_i(u))G(u,y_i(u))} \, \frac{F(u,y)}{y - y_i(u)}.$$

Then $\Psi(u,y) \in \mathbb{C}[[u]][y]$ and $H(u,y_i(u)) = \Psi(u,y_i(u))G(u,y_i(u))$ for $i = 1,\dots,n$. Therefore $H(u,y) \equiv \Psi(u,y)G(u,y) \bmod (y - y_i(u))$ for $i = 1,\dots,n$ and $H(u,y) \equiv \Psi(u,y)G(u,y) \bmod F(u,y)$ what is equivalent to $H(u,y) \in (F(u,y),G(u,y))\mathbb{C}[[u]][y]$.

Lemma 3.4. *If $\Psi(u,y) = \Psi_0(u)y^{n-1} + \cdots + \Psi_{n-1}(u) \in \mathbb{C}[[u]][y]$, then*

$$\sum_{i=1}^{n} \frac{\Psi(u,y_i(u))}{\frac{\partial F}{\partial y}(u,y_i(u))} = \Psi_0(u)$$

Proof. The lemma follows immediately from the Lagrange interpolation formula.

Proof of Theorem 3.1. We may assume that $f_i = f_i(x,y)$ are y–distinguished polynomials and (after replacing g, h by the rests of division by f) g, $h \in \mathbb{C}[[x]][y]$. We have

$$\begin{aligned}(f_i,g)_0 + c_i(f) &= (f_i,g)_0 + \mu(f_i) + \sum_{j\neq i}(f_i,f_j)_0 = (f_i,g)_0 - (f_i,x)_0 + 1 \\ &+ (f_i,\partial f_i/\partial y)_0 + \sum_{j\neq i}(f_i,f_j)_0 = (f_i,g)_0 - (f_i,x)_0 + 1 + (f_i,\partial f/\partial y)_0\end{aligned}$$

by Teissier's Lemma.

Let $n_i = (f_i, x)_0$ for $i = 1, \dots, r$. The Noether's conditions are equivalent to

$$(1) \qquad (f_i, h)_0 \geq (f_i, g)_0 + \left(f_i, \frac{\partial f}{\partial y}\right)_0 - n_i + 1 \quad \text{for } i = 1, \dots, r$$

By Puiseux's Theorem we can write

$$f_i(t^{n_i}, y) = (y - y_{i1}(t)) \dots (y - y_{in_i}(t)) \quad \text{in } \mathbb{C}[[t]][y]$$

where $y_{i1}(t), \dots, y_{in_i}(t)$ are $\mathbb{C}[[t^{n_i}]]$–conjugate i. e. $y_{ij}(t) = y_{i1}(\epsilon_j t)$ for some ϵ_j such that $\epsilon_j^{n_i} = 1$.

Thus, for every $h(x, y) \in \mathbb{C}[[x, y]]$:

$$\operatorname{ord} h(t^{n_i}, y_{i1}(t)) = \dots = \operatorname{ord} h(t^{n_i}, y_{in_i}(t)) = (f_i, h)_0$$

and we can rewrite (1) in the form
(2)

$$\operatorname{ord} h(t^{n_i}, y_{ij}(t)) \geq \operatorname{ord} g(t^{n_i}, y_{ij}(t)) + \operatorname{ord} \frac{\partial f}{\partial y}(t^{n_i}, y_{ij}(t)) - n_i + 1 \quad \text{for } i = 1, \dots, r$$

or else

$$(3) \qquad \operatorname{ord}(t^{n_i - 1} h(t^{n_i}, y_{ij}(t)) \geq \operatorname{ord}(g(t^{n_i}, y_{ij}(t)) \frac{\partial f}{\partial y}(t^{n_i}, y_{ij}(t)))$$

Let $N = n_1 \dots n_r$ and $\bar{y}_{ij}(u) = y_{ij}(u^{N/n_i})$ for $i = 1, \dots, n_i$. Obviously $\frac{N}{n_i}(n_i - 1) \leq N - 1$, therefore (3) implies

$$(4) \qquad \operatorname{ord}(u^{N-1} h(u^N, \bar{y}_{ij}(u))) \geq \operatorname{ord} g(u^N, \bar{y}_{ij}(u)) \frac{\partial f}{\partial y}(u^N, \bar{y}_{ij}(u))$$

and we can apply Lemma 3.3 to the polynomials $F(u, y) = f(u^N, y) = \prod(y - \bar{y}_{ij}(u))$, $G(u, y) = g(u^N, y)$ and $H(u, y) = u^{N-1} h(u^N, y)$. We get

$$u^{N-1} h(u^N, y) \in (f(u^N, y), g(u^N, y)) \mathbb{C}[[u]][y]$$

We have seen in the proof of Proposition 1.1 that $\mathbb{C}[[u]][y] = \sum_{i=0}^{N-1} \mathbb{C}[[u^N]][y] u^i$ is a free $\mathbb{C}[[u^N]][y]$–module, so

$$h(u^N, y) \in (f(u^N, y), g(u^N, y)) \mathbb{C}[[u^N]][y]$$

and consequently $h(x, y) \in (f(x, y), g(x, y)) \mathbb{C}[[x]][y]$.

Proof of Theorem 3.2. Suppose that there is a $\Psi = \Psi(x, y) \in \mathbb{C}[[x, y]]$ such that

$$(5) \qquad (f, \Psi)_0 = \mu(f) - 1$$

We may assume that $f = f(x, y)$ is y–distinguished polynomial with degree $n \geq 1$ and $\Psi = \Psi(x, y)\mathbb{C}[[x]][y]$ with $\deg_y \Psi \leq n - 1$. By Teissier's Lemma we can rewrite (5) in the form

$$(f, x\Psi)_0 = (f, \partial f / \partial y)_0 \tag{6}$$

By Puiseux's theorem we have $f(u^n, y) = \prod_{i=1}^n (y - y_i(u))$. We check that (6) is equivalent to

$$\operatorname{ord} u^n \Psi(u^n, y_i(u)) = \operatorname{ord} \frac{\partial f}{\partial y}(u^n, y_i(u)) \tag{7}$$

By Lemma 3.4 applied to $\Psi(u^n, y)$ and $f(u^n, y)$ we get

$$\sum_i \frac{u^n \Psi(u^n, y_i(u))}{\frac{\partial f}{\partial y}(u^n, y_i(u))} = u^n \Psi_0(u^n) \tag{8}$$

Contradiction, because the left–hand side of (8) is of order zero by (7).

4. A property of the jacobian.

We keep the notations and assumptions introduced in Section 3.

Proposition 4.1. *Let f, $g \in \mathbb{C}[[x, y]]$ be two power series without constant term, $f = f_1, \ldots, f_r$ has no multiple factors. Let $J = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$. Then*
(i) $(f_i, J)_0 = (f_i, g)_0 + c_i(f) - 1$ *for* $i = 1, \ldots, r$,
(ii) $(f, J)_0 = (f, g)_0 + \mu(f) - 1$.

Proof. Let $(t^{n_i}, y_i(t))$ be Puiseux's parametrization of the branch $f_i = 0$. Differentiating the relations $f_i(t^{n_i}, y_i(t)) = 0$ gives

$$\frac{\partial f}{\partial x}(t^{n_i}, y_i(t)) n_i t^{n_i - 1} + \frac{\partial f}{\partial y}(t^{n_i}, y_i(t)) y_i'(t) = 0$$
$$\frac{\partial g}{\partial x}(t^{n_i}, y_i(t)) n_i t^{n_i - 1} + \frac{\partial g}{\partial y}(t^{n_i}, y_i(t)) y_i'(t) = \frac{d}{dt} g(t^{n_i}, y_i(t))$$

Thus, by Cramer's identities, we get

$$n_i t^{n_i - 1} J(t^{n_i}, y_i(t)) = \left(-\frac{d}{dt} g(t^{n_i}, y_i(t))\right) \frac{\partial f}{\partial y}(t^{n_i}, y_i(t)) \tag{1}$$

Hence

$$(f_i, x)_0 - 1 + (f_i, J)_0 = (f_i, g)_0 - 1 + \left(f_i, \frac{\partial f}{\partial y}\right)_0 \tag{2}$$

On the other hand, we have

$$(3)\quad \left(f_i, \frac{\partial f}{\partial y}\right)_0 = \sum_{j\neq i}(f_i, f_j)_0 + \left(f_i, \frac{\partial f_i}{\partial y}\right)_0 = \sum_{j\neq i}(f_i, f_j)_0 + \mu(f_i) + (f_i, x)_0 - 1$$

by Teissier's lemma. Combining (2) and (3) we get (i).

Now, we check (ii):

$$\begin{aligned}(f, J)_0 &= \sum_{i=1}^{r}(f_i, J)_0 = \sum_{i=1}^{r}(f_i, g)_0 + c_i(f) - 1) \\ &= (f, g)_0 + \sum_{i=1}^{r}\left(\mu(f_i) + \sum_{j\neq i}(f_i, f_j)_0 - 1\right) \\ &= (f, g)_0 + \sum_{i=1}^{r}\mu(f_i) + 2\sum_{1\le i<j\le r}(f_i, f_j)_0 - r = (f, g)_0 + \mu(f) - 1.\end{aligned}$$

Now we can prove the following

Theorem 4.2. *Let f, $g \in \mathbb{C}[[x, y]]$ be two comprime power series. Suppose that f has no multiple factors. Then for any power series $a \in \mathbb{C}[[x, y]]$ without constant term*

$$aJ \in (f, g)\mathbb{C}[[x, y]].$$

Proof. Theorem 4.2 follows from Proposition 4.1(i) and Theorem 3.1.

Note. The assumption "f has no multiple factors" is irrelevant. In fact, if f and $g \in \mathbb{C}[[x, y]]$ have no common factor then by Bertini's theorem the power series $f + tg$ with "generic" $t \in \mathbb{C}$ is reduced.

5. Semigroup of a branch

The semigroup $\Gamma(f)$ of the branch $f = 0$ is, by definition, the set of positive integers $(f, g)_0$ as g ranges over all $g \in \mathbb{C}[[x, y]]$ such that $g \not\equiv 0 \bmod f$. Consequently $\Gamma(f)$ is a semi–group of positive integers, i. e. the sum of any two elements in $\Gamma(f)$ is in $\Gamma(f)$ and $0 \in \Gamma(f)$. The semigroup $\Gamma(f)$ is an analytic invariant of the branch $f = 0$: if ϕ is an analytic transformation then $\Gamma(f) = \Gamma(f \circ \phi)$.

Lemma 5.1. *For any branch $f = 0$: $\Gamma(f) - \Gamma(f) = \mathbb{Z}$ i. e. any integer can be written in the form $(f, \phi)_0 - (f, \psi)_0$ with some ϕ, $\psi \in \mathbb{C}[[x, y]]$.*

Proof. The Puiseux's parametrization $p(t) = (t^n, y(t))$ of $f(x, y) = 0$ is primitive, thus $\mathbb{C}((t^n, y(t))) = \mathbb{C}((t))$ (cf. [8]. chapter 12). Let us fix an arbitrary $a \in \mathbb{Z}$. Then $t^a = \phi(t^n, y(t))/\psi(t^n, y(t))$ for some ϕ, $\psi \in \mathbb{C}[[x, y]]$ and taking orders gives $a = (f, \phi)_0 - (f, \psi)_0$.

We illustrate the results from Section 3 by proving the following basic property of $\Gamma(f)$:

Theorem 5.2. *The semigroup $\Gamma(f)$ contains all integers greater or equal to the Milnor number $\mu(f)$. The number $\mu(f)-1$ does not belong to $\Gamma(f)$.*

Proof. Let p be an integer such that $p \geq \mu(f)$. By Lemma 5.1 we can write $p = (f,\phi)_0 - (f,\psi)_0$ for some ϕ, $\psi \in \mathbb{C}[[x,y]]$. Then $(f,\phi)_0 = (f,\psi)_0 + p \geq (f,\psi)_0 + \mu(f)$ and by Noether's Theorem $\phi = Af + B\psi$ for some A, $B \in \mathbb{C}[[x,y]]$. Thus $p = (f, Af+B\psi)_0 - (f,\psi)_0 = (f,B)_0$ and we are done.

The second part of 5.2 follows immediately from 3.2.

Theorem 5.2 has an interesting algebraical meaning. Suppose that $f \in \mathbb{C}[[x,y]]$ is irreducible and let us consider its local ring $\mathcal{O}_f$, the field of fractions $\mathcal{M}_f$ and the normalisation $\widehat{\mathcal{O}}_f$. Note that if $p(t) = (t^n, y(t))$ is Puiseux's parametrization, then $\mathcal{O}_f = \mathbb{C}[[t^n, y(t)]]$, $\widehat{\mathcal{O}}_f = \mathbb{C}[[t]]$ and $\mathcal{M}_f = \mathbb{C}((t))$.

Let us define $\nu_f : \mathcal{M}_f \to \mathbb{Z}$ by putting $\nu_f(\phi/\psi) = (f,\phi)_0 - (f,\psi)_0$. Then ν_f is a valuation of $\mathcal{M}_f$, $\nu_f(\widehat{\mathcal{O}}_f) = \mathbb{N}$ and $\nu_f(\mathcal{O}_f) = \Gamma(f)$. Using Theorem 5.2 one sees that the conductor $\mathcal{O}_f : \widehat{\mathcal{O}}_f$ equals to $\{\xi \in \mathcal{M}_f : \nu_f(\xi) \geq \mu(f)\}$ and we get the following formula for the Milnor number:

$$\mu(f) = \dim_{\mathbb{C}} \frac{\widehat{\mathcal{O}}_f}{\mathcal{O}_f : \widehat{\mathcal{O}}_f}$$

The reader interested in arithmetical methods is referred to S. S. Abhyankar's lectures [1] and [2].

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Liczba Milnora płaskiej krzywej algebroidalnej

Streszczenie. W pracy podany jest elementarny wykład podstawowych własności liczby Milnora. W oparciu o lokalny wariant twierdzenia Noethera udowodniona jest formuła wiążąca liczbę Milnora z przewodnikiem pierścienia lokalnego krzywej.

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