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THE FUKUI INEQUALITY FOR THE ŁOJASIEWICZ EXPONENT OF NONDEGENERATE CONVENIENT SINGULARITIES

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Abstract

In the article we give a new elementary proof of the Fukui inequality [F] for the Łojasiewicz exponent of nondegenerate singularities with convenient Newton diagrams. In the proof we use only the Curve Selection Lemma.

1 Introduction

Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function in an open neighborhood of $0 \in \mathbb{C}^n$ and $\sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu$ be the Taylor expansion of f at 0. We define $\Gamma_+(f) := \mathrm{conv}\{\nu + \mathbb{R}^n_+ : a_\nu \neq 0\} \subset \mathbb{R}^n$ and call it *the Newton diagram* of f. Let $u \in \mathbb{R}^n_+ \setminus \{0\}$. Put $l(u, \Gamma_+(f)) := \inf\{< u, v >: v \in \Gamma_+(f)\}$ and $\Delta(u, \Gamma_+(f)) := \{v \in \Gamma_+(f) :< u, v >= l(u, \Gamma_+(f))\}$. We say that $S \subset \mathbb{R}^n$ is a *face* of $\Gamma_+(f)$, if $S = \Delta(u, \Gamma_+(f))$ for some $u \in \mathbb{R}^n_+ \setminus \{0\}$. The vector u is called *the primitive vector* of S. It is easy to see that S is a closed and convex set and $S \subset \mathrm{Fr}(\Gamma_+(f))$, where $\mathrm{Fr}(A)$ denotes the boundary of A. One can prove that a face $S \subset \Gamma_+(f)$ is compact if and only if all coordinates of its primitive vector u are positive. We call the family of all compact faces of $\Gamma_+(f)$ the *Newton boundary* of f and denote by $\Gamma(f)$. We denote by $\Gamma^k(f)$ the set of all compact k-dimensional faces of $\Gamma(f)$, $k = 0, \ldots, n-1$. For every compact face $S \in \Gamma(f)$ we define quasihomogeneous polynomial $f_S := \sum_{\nu \in S} a_\nu z^\nu$. We say that f is *nondegenerate on the face* $S \in \Gamma(f)$, if the system of equations $\frac{\partial f_S}{\partial z_1} = \ldots = \frac{\partial f_S}{\partial z_n} = 0$ has no solution in $(\mathbb{C}^*)^n$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We say that f is *nondegenerate in the Kouchnirenko's sense* (shortly *nondegenerate*) if it is nondegenerate on each face of $\Gamma(f)$. We say that f is a *singularity* if f is a nonzero holomorphic function in some open neighborhood of the origin and $f(0) = 0$, $\nabla f(0) = 0$, where $\nabla f = (f'_{z_1}, \ldots, f'_{z_n})$. We say that f is *an isolated singularity* if f is a singularity, which has an isolated critical point in the origin i.e. additionally $\nabla f(z) \neq 0$ for $z \neq 0$.

Let $i \in \{1, \ldots, n\}$, $n \geq 2$.

Definition 1.1 *We say that $S \in \Gamma^{n-1}(f) \subset \mathbb{R}^n$ is an exceptional face with respect to the axis OX_i if one of its vertices is at distance 1 to the axis OX_i and another vertices constitute $(n-2)$-dimensional face which lies in one of the coordinate hyperplane including the axis OX_i.*

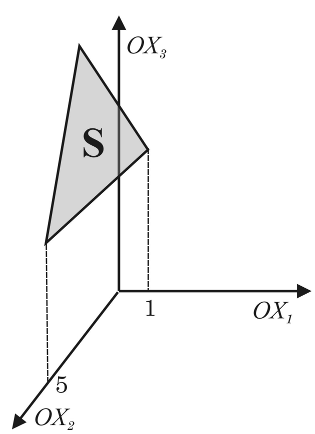

Figure 1: An exceptional face S with respect to the axis OX_3.

We say that $S \in \Gamma^{n-1}(f)$ is *an exceptional face of* f if there exists $i \in \{1, \ldots, n\}$ such that S is an exceptional face with respect to the axis OX_i. Denote by E_f the set of exceptional faces of f.

Definition 1.2 *We say that the Newton diagram of f is convenient if it has nonempty intersection with every coordinate axis.*

Definition 1.3 *We say that the Newton diagram of f is nearly convenient if its distance to every coordinate axis doesn't exceed 1.*

For every $(n-1)$-dimensional compact face $S \in \Gamma(f)$ we shall denote by $x_1(S), \ldots, x_n(S)$ coordinates of intersection of the hyperplane determined by face S with the coordinate axes. We define $m(S) := \max\{x_1(S), \ldots, x_n(S)\}$. It is easy to see that

$$x_i(S) = \frac{l(u, \Gamma_+(f))}{u_i},\ i = 1, \ldots, n,$$

where u is a primitive vector of S. It is easy to check that the Newton diagram $\Gamma_+(f)$ of an isolated singularity f is nearly convenient. So, "nearly convenience" of the Newton diagram is a neccesary condition for f to be an isolated singularity. For a singularity f such that $\Gamma^{n-1}(f) \neq \emptyset$, we define

(1)
$$m_0(f) := \max_{S \in \Gamma^{n-1}(f)} m(S).$$

It is easy to see that in the case $\Gamma_+(f)$ is convenient $m_0(f)$ is equal to the maximum of coordinates of the points of the intersection of the Newton diagram of f and the union of all axes.

Remark 1.4 *A definition of $m_0(f)$ for all singularities (even for $\Gamma^{n-1}(f) = \emptyset$), can be found in* [F]*. In the case $\Gamma^{n-1}(f) \neq \emptyset$ both definitions are equivalent.*

Let $f = (f_1, \ldots, f_n) : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ be a holomorphic mapping having an isolated zero at the origin. We define the number

(2)
$$l_0(f) := \inf\{\alpha \in \mathbb{R}_+ : \exists_{C>0} \exists_{r>0} \forall_{\|z\|<r} \|f(z)\| \geq C\|z\|^\alpha\}$$

and call it *the Łojasiewicz exponent* of the mapping f. There are formulas and estimations of the number $l_0(f)$ under some nondegeneracy conditions of f (see [B], [BE1], [Lt], [Ph]).

Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be an isolated singularity. We define a number $\pounds_0(f) := l_0(\nabla f)$ and call it *the Łojasiewicz exponent of singularity f*. Now we give some important known properities of the Łojasiewicz exponent (see [L-JT]):

(a) $\pounds_0(f)$ is a rational number.

(b) $\pounds_0(f) = \sup\{\frac{\mathrm{ord}\, \nabla f(z(t))}{\mathrm{ord}\, z(t)} : 0 \neq z(t) \in \mathbb{C}\{t\}^n,\ z(0) = 0\}$.

(c) The infimum in the definition of the Łojasiewicz exponent is attained for $\alpha = \pounds_0(f)$.

(d) $s(f) = [\pounds_0(f)] + 1$, where $s(f)$ is *the degree of C^0-sufficiency of f* [ChL].

Lenarcik gave in [L] the formula for the Łojasiewicz exponent for singularities of two variables, nondegenerate in Kouchnirenko sense, in terms of its Newton diagram (another formulas in general two-dimensional case see [CK1], [CK2]).

Theorem 1.5 ([L]) *Let* $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ *be an isolated nondegenerate singularity and* $\Gamma^1(f) \setminus E_f \neq \emptyset$. *Then*

$$\pounds_0(f) = \max_{S \in \Gamma^1(f) \setminus E_f} m(S) - 1. \tag{3}$$

Remark 1.6 *In two-dimensional case one can prove that for isolated singularities such that* $\Gamma^1(f) \setminus E_f = \emptyset$, *i.e.* $\Gamma^1(f)$ *consist of only exceptional segments, we have* $\pounds_0(f) = 1$.

Let us pass to three dimensional case. Denote by $\overline{AB}$ the segment joining two different points $A, B \in \mathbb{R}^3$. We consider the following segments in $\mathbb{R}^3$:

$$I_1^k = \overline{(0,1,1)(k,0,0)},\ I_2^k = \overline{(1,0,1)(0,k,0)},\ I_3^k = \overline{(1,1,0)(0,0,k)},\ k \in \{2, 3 \ldots\}.$$

Put $\mathcal{J} := \{I_j^k : j = 1,2,3,\ k = 2,3,\ldots\}$. Every segment I of this family intersects exactly one coordinate axis in exactly one point. We denote by $m(I)$ nonzero coordinate of this point (equal to k). We have the following result.

Theorem 1.7 ([O1]) *Let* $f : \left(\mathbb{C}^3, 0\right) \longrightarrow (\mathbb{C}, 0)$ *be an isolated and nondegenerate singularity.*

1^0 *If* $\Gamma^2(f) = \emptyset$ *or* $\Gamma^2(f) = E_f$, *then there exists excatly one segment* $I \in \mathcal{J} \cap \Gamma^1(f)$ *and*

$$\pounds_0(f) = m(I) - 1.$$

2^0 *If* $\Gamma^2(f) \setminus E_f \neq \emptyset$, *then*

$$\pounds_0(f) \leq \max_{S \in \Gamma^2(f) \setminus E_f} m(S) - 1. \tag{4}$$

Now we pass to n-dimensional case. In multidimensional case we have an upper bounds for $\pounds_0(f)$, which was given by T. Fukui in 1991 without removing any exceptional faces (see also [A],[O],[O1]). It is similar to the one given in Theorem 1.7 2^0 but we conjecture that in the inequality (5) after removing exceptional faces there is the equality. It was proved to be true for quasihomogeneous surface singularities in [KOP].

Theorem 1.8 ([F]) *Let* $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ *be an isolated nondegenerate singularity. Then*

$$\pounds_0(f) \leq m_0(f) - 1. \tag{5}$$

The proof of the above theorem is technically intricate. We prove this theorem in an elementary way in the case Newton diagram of f is convenient. Precisely we prove, using only the Curve Selection Lemma, the following theorem.

Theorem 1.9 *Let* $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 2$, *be an isolated nondegenerate singularity such that* $\Gamma(f)$ *is convenient. Then*

$$\pounds_0(f) \leq \max_{S \in \Gamma^{n-1}(f)} m(S) - 1. \tag{6}$$

Remark 1.10 *It is easy to see that if* $\Gamma(f)$ *is convenient then* $\Gamma^{n-1}(f) \neq \emptyset$.

2 Proof of the Theorem 1.9

We give now a lemma used in the proof.

Lemma 2.1 *Let* $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 3$, *be a holomorphic function in an open neighborhood of* $0 \in \mathbb{C}^n$ *and* $g(z_1, \ldots, z_k) := f(z_1, \ldots, z_k, 0, \ldots, 0) \neq 0$, $k \geq 2$. *Then*

$$\Gamma(g) = \{S \in \Gamma(f) : S \subset \{x_{k+1} = \ldots = x_n = 0\}\}. \tag{7}$$

Proof." $\subset$ ". Let $S \in \Gamma(g)$, so $S = \Delta(u, \Gamma_+(g))$ for some $u \in (\mathbb{R}_+ \setminus \{0\})^k$. Of course $S \subset \Gamma_+(f) \cap \{x_{k+1} = \ldots = x_n = 0\}$. Set $u' = (u_1, \ldots, u_k, l(u, \Gamma_+(g)) + 1, \ldots, l(u, \Gamma_+(g)) + 1) \in \mathbb{R}^n$. We show that $S = \Delta(u', \Gamma_+(f))$. By definition of u' we have that $l(u', \Gamma_+(f))$ can be realised only for $v \in \Gamma_+(f) \cap \{x_{k+1} = \ldots = x_n = 0\}$. But it is easy to check that $\Gamma_+(f) \cap \{x_{k+1} = \ldots = x_n = 0\} = \Gamma_+(g)$. So we get $l(u', \Gamma_+(f)) = l(u, \Gamma_+(g))$ and $\Delta(u', \Gamma_+(f)) = \Delta(u, \Gamma_+(g))$. Reasumming $S = \Delta(u', \Gamma_+(f))$, it is in $\Gamma(f)$.

" $\subset$ ". Let $S \in \Gamma(f)$ i $S \subset \{x_{k+1} = \ldots = x_n = 0\}$. Then $S = \Delta(u, \Gamma_+(f))$ for some $u \in (\mathbb{R}_+ \setminus \{0\})^n$ and as we observed above $\Gamma_+(f) \cap \{x_{k+1} = \ldots = x_n = 0\} = \Gamma_+(g)$. So $l(u, \Gamma_+(f)) = l(u', \Gamma_+(g))$, where $u' = (u_1, \ldots, u_k)$. It follows that $\Delta(u', \Gamma_+(g)) = \Delta(u, \Gamma_+(f))$ and $S \in \Gamma(g)$. That concludes the proof. ■

We can go to the proof of Theorem 1.9.

Proof. It is enough to show that

$$|\nabla f(x)| \geq C|x|^{m_0(f)-1}.$$

for some $C > 0$ in some neighborhood of $0 \in \mathbb{C}^n$. Suppose to the contrary that this inequality isn't true. Hence be the Curve Selection Lemma we get that there exists an analytic curve $\phi : [0, \epsilon) \to (\mathbb{C}^n, 0)$ such that

$$\operatorname{ord} |\nabla f \circ \phi(t)| > \operatorname{ord} |\phi(t)|^{m_0(f)-1} \tag{8}$$

Let $J = \{j \in \{1, \ldots n\} : \phi_j \neq 0\}$. We have

$$\phi_j(t) = x_j^0 t^{q_j} + \textit{higher order terms},\ j \in J$$

for some $q_j > 0$, $x_j^0 \neq 0$. Set $q_* = \min_{j \in J} q_j$ and let $\Gamma' := \Gamma_+(f) \cap \mathbb{R}^J$, where $\mathbb{R}^J \subset \mathbb{R}^n$ is the linear subspace of $\mathbb{R}^n$ spanned by the axis OX_j, $j \in J$. Then vector $w = (\operatorname{ord} \phi_j)_{j \in J}$ supports a compact face S of Γ' and

$$\frac{d}{q_*} \leq m_0(f), \tag{9}$$

where $\sum_{j \in J} q_j x_j = d$ is the equation of the supporting hyperplane of face S. Moreover by Lemma 2.1 $S \in \Gamma(f)$. We get further

$$f'_{z_i} \circ \phi(t) = t^{d-q_i} \operatorname{in}_w f'_{z_i}(x_1^0, \ldots x_n^0) + \textit{ higher order terms},\ i = 1, 2, \ldots n$$

where $x_j^0 := 1$ for $j \notin J$. There exists a variable z_j, $j \in J$, which appears in a monomial of f_S with non-zero coefficient. For these variables we have $\mathrm{in}_w f'_{z_i} = (f_S)'_{z_i}$. Since f is nondegenerate on the face S, so among these variables there exists a variable z_{j_0} such that $(f_S)'_{z_{j_0}}(x_0^1, \ldots, x_0^n) \neq 0$. Then $\mathrm{ord}(f'_{z_{j_0}} \circ \phi(t)) = d - q_{j_0}$ and by inequality (8) we get $d - q_{j_0} > q_*(m_0(f) - 1)$. Hence after easy transformations we get

$$\frac{d}{q_*} > m_0(f) + \left(\frac{q_{j_0}}{q_*} - 1\right),$$

which contradicts inequality (9). It finishes the proof. ■

Example 2.2 *Let $f(z_1, z_2, z_3) := z_3^{20} + z_1^3 + z_2^3 + z_3^4 z_1 + z_3^4 z_2$. It is easy to check that f is an isolated nondegenerate singularity. We also see that $\Gamma(f)$ is convenient and consists of two faces S_1 and S_2 but the face $S_1 = \mathrm{conv}\{(1,0,4), (0,1,4), (0,0,20)\}$ is an exceptional with respect to the axis OX_3 and $m_0(f) = m(S_1) = 20$ (see Fig. 2).*

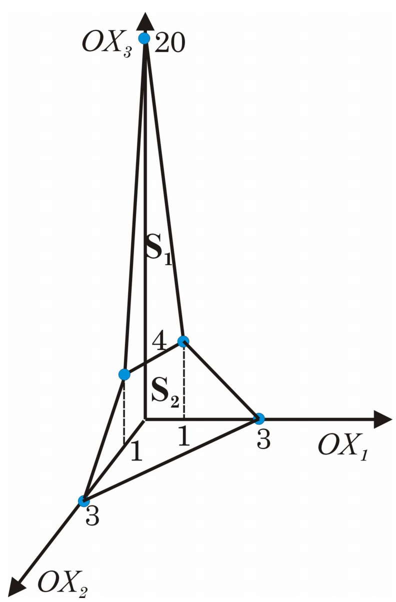

Figure 2: The Newton boundary of singularity in Example 2.2

By Theorem 1.9

$$Ł_0(f) \leq m_0(f) - 1 = 20 - 1 = 19,$$

and by Theorem 1.7 we get that

$$Ł_0(f) \leq \max_{S \in \Gamma^2(f) \setminus \{S_1\}} m(S) - 1 = m(S_2) - 1 = 6 - 1 = 5. \tag{10}$$

Hence the last estimation is better. It is easy to check that singularity $g := f - z_3^{20}$ *is an isolated and weighted with weights* 3, 3, 6. *Hence by Theorem 1 of paper* [KOP] *we have*

$$\mathcal{L}_0(g) = \max\{3, 3, 6\} - 1 = 5.$$

Since $\operatorname{ord}(\nabla f - \nabla g) > \mathcal{L}_0(g)$, *then by Lemma 1.4 in* [P] *we get that* $\mathcal{L}_0(f) = \mathcal{L}_0(g) = 5$. *Hence estimation obtained by Theorem 1.7 is optimal.*

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Nierówność Fukui dla wykładnika Łojasiewicza niezdegenerowanych dogodnych osobliwości

Streszczenie. W pracy podajemy nowy elementarny dowód nierówności Fukui [F] na wykładnik Łojasiewicza osobliwości niezdegenerowanych o dogodnych diagramach Newtona. W tym dowodzie korzystamy tylko z Lematu o Wyborze Krzywej.

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