A closedness theorem in geometry over Henselian valued fields

XL Conference and Workshop
"Analytic and Algebraic Geometry"
January 07-11, 2019, Łódź

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Aim of the talk

The aim is to present geometry of algebraic subvarieties of K^n over arbitrary Henselian valued fields K of equicharacteristic zero, developed in my recent papers [N2, N3, N4, N5]. At the center of my approach is my closedness theorem that the projections $K^n \times \mathbb{P}^m(K) \to K^n$ are definably closed maps. It enables, in particular, application of resolution of singularities in much the same way as over locally compact ground fields. The basic tools involved are quantifier elimination for Henselian valued fields (in the language $\mathcal{L}$ of Denef–Pas) and preparation cell decomposition due to Pas as well as relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts) due to Cluckers–Halupczok.

In the proof of the closedness theorem I apply i.a. the local behaviour of definable functions of one variable and fiber shrinking, being a relaxed version of curve selection. To achieve the former result over arbitrary Henselian valued fields, I first examine functions given by algebraic power series.

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As applications of the closedness theorem, I achieve several non-Archimedean versions of the Łojasiewicz inequality (by two instances of quantifier elimination indicated above and the closedness theorem) and of curve selection (by resolution of singularities and the closedness theorem), extending continuous hereditarily rational functions (cf. [K-N] for the real algebraic version) as well as the theory of regulous functions, sets and sheaves, including regulous versions of Nullstellensatz and Cartan's theorems A and B (cf. [FHMM] for the real algebraic versions).

Examples of Henselian valued fields

Examples of such fields are the quotient fields of the rings of formal power series and of Puiseux series with coefficients from a field $\mathbb{k}$ of characteristic zero as well as the fields of Hahn series (maximally complete valued fields also called Malcev–Neumann fields):

$$\mathbb{k}((t^{\Gamma})) := \left\{ f(t) = \sum_{\gamma \in \Gamma} a_{\gamma} t^{\gamma} : \ a_{\gamma} \in \mathbb{k}, \ \operatorname{supp} f(t) \text{ is well ordered} \right\}.$$

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Geometry over algebraically closed or real closed valued fields K is much tamer than in the general case, though it does not have definable Skolem functions either. For the real closed valued fields, cells are defined — similarly as for o-minimal structures — by equalities and inequalities, but applied to definable, not necessarily continuous functions with values in the Dedekind completion of K.

Closedness Theorem

Other, new applications of the closedness theorem are piecewise continuity of definable functions, Hölder continuity of definable functions on closed bounded subsets of K^n and a non-Archimedean, definable version of the Tietze–Urysohn extension theorem from [N5]. In the paper [N4], I established a non-Archimedean version of the closedness theorem over Henselian valued fields with analytic structure along with several applications.

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Theorem

Let D be an $\mathcal{L}$-definable subset of K^n. Then the canonical projection

$$\pi : D \times R^m \longrightarrow D$$

is definably closed in the K-topology, i.e. if $B \subset D \times R^m$ is an $\mathcal{L}$-definable closed subset, so is its image $\pi(B) \subset D$.

Corollary

Let A be a closed $\mathcal{L}$-definable subset of R^m or of $\mathbb{P}^m(K)$. Then every continuous $\mathcal{L}$-definable map $f : A \to K^n$ is definably closed in the K-topology.

Corollary

Let A be a closed $\mathcal{L}$-definable subset of R^m or of $\mathbb{P}^m(K)$. Then every continuous $\mathcal{L}$-definable map $f : A \to K^n$ is definably closed in the K-topology.

Corollary

Let X be a smooth K-variety, ϕ_i, $i = 0, \ldots, m$, regular functions on X, D be an $\mathcal{L}$-definable subset of $X(K)$ and $\sigma : Y \longrightarrow X$ the blow-up of the ideal $(\phi_0, \ldots, \phi_m)$. Then the restriction

$$\sigma : Y(K) \cap \sigma^{-1}(D) \longrightarrow D$$

is a definably closed quotient map.

Descent principle for blow-ups

Corollary

Under the assumptions of the above corollary, every continuous $\mathcal{L}$-definable function

$$g : Y(K) \cap \sigma^{-1}(D) \longrightarrow K$$

that is constant on the fibers of the blow-up σ descends to a (unique) continuous $\mathcal{L}$-definable function $f : D \longrightarrow K$.

Implicite function theorem

Below we present the version of the implicite function theorem from [N3] (see also [P-Z, Kuhl, G-G-MB]).

Let $(R, \mathfrak{m})$ be a Henselian ring, $0 \leqslant r < n$, $p = (p_{r+1}, \ldots, p_n)$ be an $(n-r)$-tuple of polynomials $p_{r+1}, \ldots, p_n \in R[X]$, $X = (X_1, \ldots, X_n)$, and

$$J := \frac{\partial(p_{r+1}, \ldots, p_n)}{\partial(X_{r+1}, \ldots, X_n)}, \quad e := J(\mathbf{0}).$$

Suppose that

$$\mathbf{0} \in V := \{x \in R^n : p_{r+1}(x) = \ldots = p_n(x) = 0\}.$$

Theorem

If $e \neq 0$*, then there exists a unique continuous map*

$$\phi : (e^2 \cdot \mathfrak{m})^{\times r} \longrightarrow (e \cdot \mathfrak{m})^{\times (n-r)}$$

which is definable in the language of valued fields and such that $\phi(0) = 0$ *and the graph map*

$$(e^2 \cdot \mathfrak{m})^{\times r} \ni u \longrightarrow (u, \phi(u)) \in (e^2 \cdot \mathfrak{m})^{\times r} \times (e \cdot \mathfrak{m})^{\times (n-r)}$$

is an open embedding into the zero locus V *of the polynomials* p *and, more precisely, onto*

$$V \cap \left[(e^2 \cdot \mathfrak{m})^{\times r} \times (e \cdot \mathfrak{m})^{\times (n-r)} \right].$$

.

The non-Archimedean Artin–Mazur theorem from [N3] (see cf. [AM, BCR] for the classical versions).

Theorem

Let $\phi = \phi_1 \in (X)K[[X]]$ be an algebraic formal power series. Then there exist polynomials

$$p_1, \dots, p_r \in K[X, Y], \quad Y = (Y_1, \dots, Y_r),$$

and formal power series $\phi_2, \dots, \phi_r \in K[[X]]$ such that

$$e := \frac{\partial(p_1, \dots, p_r)}{\partial(Y_1, \dots, Y_r)}(\mathbf{0}) = \det \left[\frac{\partial p_i}{\partial Y_j}(\mathbf{0}) : \ i, j = 1, \dots, r \right] \neq 0,$$

and

$$p_i(X_1, \dots, X_n, \phi_1(X), \dots, \phi_r(X)) = 0, \quad i = 1, \dots, r.$$

Functions given by algebraic power series

Corollary

Let $\phi \in (X)K[[X]]$ be an algebraic power series with irreducible polynomial $p(X,T) \in K[X,T]$. Then there is an $a \in K$, $a \neq 0$, and a unique continuous function

$$\widetilde{\phi} : a \cdot R^n \longrightarrow K$$

which is definable in the language of valued fields and such that $\widetilde{\phi}(0) = 0$ and $p(x, \widetilde{\phi}(x)) = 0$ for all $x \in a \cdot R^n$. □

A non-archimedean Abhyankar–Jung theorem

Let K be an algebraically closed field of characteristic zero. Consider a henselian $K[X]$-subalgebra $K\langle X\rangle$ of the formal power series ring $K[[X]]$ which is closed under reciprocal (whence it is a local ring), power substitution and division by a coordinate. For a positive integer r put (cf. [N1]):

$$K\langle X^{1/r}\rangle = K\langle X_1^{1/r}, \dots, X_n^{1/r}\rangle := \left\{ a(X_1^{1/r}, \dots, X_n^{1/r}) : a(X) \in K\langle X\rangle \right\}.$$

Theorem

Every quasiordinary (i.e. whose discriminant is a normal crossing) polynomial

$$f(X;T) = T^s + a_{s-1}(X)T^{s-1} + \cdots + a_0(X) \in K\langle X\rangle[T]$$

has all its roots in $K\langle X^{1/r}\rangle$ *for some* $r \in \mathbb{N}$*; actually, one can take* $r = s!$.

A non-archimedean Newton–Puiseux theorem

Corollary

Every polynomial

$$f(X;T) = T^s + a_{s-1}(X)T^{s-1} + \cdots + a_0(X) \in K\langle X\rangle[T]$$

in one variable X has all its roots in $K\langle X^{1/r}\rangle$ for some $r \in \mathbb{N}$; one can take $r = s!$. Equivalently, the polynomial $f(X^r, T)$ splits into T-linear factors. If $f(X, T)$ is irreducible, then $r = s$ will do and

$$f(X^s, T) = \prod_{i=1}^{s} (T - \phi(\epsilon^i X)),$$

where $\phi(X) \in K\langle X\rangle$ and ϵ is a primitive root of unity.

Due to the finitary character of their proofs, if the field K is not algebraically closed, the theorems of Abhyankar–Jung and Newton–Puiseux remain valid for the Henselian subalgebra $\overline{K} \otimes_K K\langle X \rangle$ of $\overline{K}[[X]]$; here $\overline{K}$ is the algebraic closure of K. The case of the algebra of algebraic power series will be applied in our analysis of definable functions of one variable.

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Theorem

Let $f : A \to K$ be an $\mathcal{L}$-definable function on a subset A of K and suppose 0 is an accumulation point of A.

Definable functions of one variable

Theorem

Then there is a finite partition of A into $\mathcal{L}$-definable sets $A_1, \ldots, A_r$ and points $w_1 \ldots, w_r \in \mathbb{P}^1(K)$ such that

$$\lim_{x \longrightarrow 0} f|A_j(x) = w_j \quad \textit{for} \;\; j = 1, \ldots, r.$$

Moreover, there is a neighbourhood U of 0 such that each definable set

$$\{(v(x), v(f(x))) : \; x \in (A_j \cap U) \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \{\infty\}), \; j = 1, \ldots, r,$$

is contained in an affine line $l = \frac{p_j}{q} \cdot k + \beta_j$, $j = 1, \ldots, r$, with rational slope, $p_j, q \in \mathbb{Z}$, $q > 0$, $\beta_j \in \Gamma$, or in $\Gamma \times \{\infty\}$.

QE for Henselian valued fields

The field K is considered in the 3-sorted language $\mathcal{L}$ of Denef-Pas; main sort: the valued field K with valuation

$$v : K \to \Gamma \cup \{\infty\};$$

two auxiliary sorts: the value group Γ and the residue field $\Bbbk$.

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The only symbols of $\mathcal{L}$ connecting the sorts are: the valuation map v and the angular component map $\overline{ac} : K \to \Bbbk$ which is multiplicative, sends 0 to 0 and coincides with the residue map on units of the valuation ring R of K.

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Theorem

([Pas]) If the valued field K is Henselian and of equicharacteristic zero, then $(K, \Gamma, \Bbbk)$ admits elimination of K-quantifiers in the language $\mathcal{L}$.

Cells

Consider an $\mathcal{L}$-definable subset D of $K^n \times \Bbbk^m$, three $\mathcal{L}$-definable funkctions

$$a(x, \xi), b(x, \xi), c(x, \xi) : D \longrightarrow K$$

and a positive integer ν. For $\xi \in \Bbbk^m$ put

$$C(\xi) := \{(x, y) \in K_x^n \times K_y : \ (x, \xi) \in D,$$

$$v(a(x, \xi)) \lhd_1 v((y - c(x, \xi))^\nu) \lhd_2 v(b(x, \xi)), \ \overline{ac}(y - c(x, \xi)) = \xi_1\},$$

where $\lhd_1, \lhd_2$ stand for $<, \leqslant$ or no condition.

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where $\lhd_1, \lhd_2$ stand for $<, \leqslant$ or no condition.

Definition. If the sets $C(\xi)$, $\xi \in \Bbbk^m$, are pairwise disjoint, the union

$$C := \bigcup_{\xi \in \Bbbk^m} C(\xi)$$

is called a cell in $K^n \times K$ with parameters ξ and center $c(x,\xi)$; $C(\xi)$ is called a fiber of the cell C.

Preparation Theorem

Theorem

([Pas]) Let $f_1(x,y),\dots,f_r(x,y)$ *be polynomials in one variable* y *with coefficients being* $\mathcal{L}$*-definable functions on* K^n_x*. Then* $K^n \times K$ *admits a finite partition into cells such that on each cell* C *with parameters* ξ *and center* $c(x,\xi)$ *and for all* $i=1,\dots,r$ *we have:*

$$v(f_i(x,y)) = v\left(\tilde{f}_i(x,\xi)(y-c(x,\xi))^{\nu_i}\right),$$

$$\overline{ac}\, f_i(x,y) = \xi_{\mu(i)},$$

where $\tilde{f}_i(x,\xi)$ *are* $\mathcal{L}$*-definable functions,* $\nu_i \in \mathbb{N}$ *for all* $i=1,\dots,r$*, and the map* $\mu : \{1,\dots,r\} \to \{1,\dots,m\}$ *does not depend on* x, y, ξ*.*

QE for ordered abelian groups

It is well known that archimedean ordered abelian groups admit quantifier elimination in the Presburger language. Much more complicated is quantifier elimination for non-archimedean groups (especially of infinite rank), going back as far as Gurevich [Gur].

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He established a transfer of sentences from ordered abelian groups to so-called coloured chains (i.e. linearly ordered sets with additional unary predicates), enhanced later to allow arbitrary formulas (his doctoral dissertation "The decision problem for some algebraic theories", Sverdlovsk, 1968), and next also by Schmitt (habilitation thesis "Model theory of ordered abelian groups", Heidelberg, 1982; see also [Sch]).

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This is a kind of relative quantifier elimination, which allowed them [Gur-Sch] in their study of the NIP property to lift model theoretic properties from ordered sets to ordered abelian groups.

Instead Cluckers–Halupczok [C-H] introduce a suitable many-sorted language $\mathcal{L}_{qe}$ with main group sort Γ and auxiliary imaginary sorts (parameterizing certain convex $\mathcal{L}$-definable subgroups), which carry the structure of a linearly ordered set with some additional unary predicates. They provide quantifier elimination relative to the auxiliary sorts, where each definable set in the group sort is a union of a family of quantifier free definable sets with parameter running a definable (with quantifiers) set of the auxiliary sorts.

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Fortunately, sometimes it is possible to directly deduce information about ordered abelian groups without any deeper knowledge of the auxiliary sorts. This may be illustrated by their theorem on piecewise linearity of definable functions [C-H, Corollary 1.10] as well as by our application of quantifier elimination in the proofs of fiber shrinking and of the closedness theorem.

Fiber shrinking

Let A and E be $\mathcal{L}$-definable subsets of K^n and K with accumulation points $a = (a_1, \dots, a_n) \in K^n$ and a_1, respectively. We call an $\mathcal{L}$-definable family of sets

$$\Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A$$

an $\mathcal{L}$-definable x_1-fiber shrinking for the set A at a if

$$\lim_{t \longrightarrow a_1} \Phi_t = (a_2, \dots, a_n),$$

i.e. for any neighbourhood U of $(a_2, \dots, a_n) \in K^{n-1}$, there is a neighbourhood V of $a_1 \in K$ such that $\emptyset \neq \Phi_t \subset U$ for every $t \in V \cap E$, $t \neq a_1$. This is a relaxed version of curve selection.

Theorem

Every $\mathcal{L}$-definable subset A of K^n with accumulation point $a \in K^n$ has, after a permutation of the coordinates, an $\mathcal{L}$-definable x_1-fiber shrinking at a.

The Łojasiewicz inequalities

Theorem

Let U and F be two $\mathcal{L}$-definable subsets of K^m, suppose U is open and F closed in the K-topology and consider two continuous $\mathcal{L}$-definable functions $f, g : A \to K$ on the locally closed subset $A := U \cap F$ of K^m. If

$$\{x \in A : g(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a continuous $\mathcal{L}$-definable function h on A such that $f^s(x) = h(x) \cdot g(x)$ for all $x \in A$.

Theorem

Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a locally closed subset A of K^n and $g : \mathcal{D}(f) \to K$ a continuous $\mathcal{L}$-definable function. Then $f^s \cdot g$ extends, for $s \gg 0$, by zero through the set $\mathcal{Z}(f)$ to a (unique) continuous $\mathcal{L}$-definable function on A.

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Theorem

Let $f, g_1, \ldots, g_m : A \to K$ be continuous $\mathcal{L}$-definable functions on a closed (in the K-topology) bounded subset A of K^m. If

$$\{x \in A : g_1(x) = \ldots = g_m(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a constant $\beta \in \Gamma$ such that

$$s \cdot v(f(x)) + \beta \geqslant v(g_1(x), \ldots, g_m(x)) \quad \textit{for all} \quad x \in A.$$

The proofs make use of elimination of valued field quantifiers, relative quantifier elimination for ordered abelian groups and the closedness theorem, whereby the problem comes down to some problems of piecewise linear geometry.

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Corollary

(Hölder continuity) Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a closed bounded subset $A \subset K^n$. Then f is Hölder continuous with a positive integer s and a constant $\beta \in \Gamma$, i.e.

$$s \cdot v(f(x) - f(z)) + \beta \geqslant v(x - z)$$

for all $x, z \in A$. In particular, f is uniformly continuous.

Curve selection

By a (valuative) semialgebraic subset of K^n we mean a (finite) Boolean combination of elementary (valuative) semialgebraic subsets, i.e. sets of the form

$$\{x \in K^n : \ v(f(x)) \leqslant v(g(x))\},$$

where f and g are regular functions on K^n. We call a map φ semialgebraic if its graph is a semialgebraic set.

Curve selection

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where f and g are regular functions on K^n. We call a map φ semialgebraic if its graph is a semialgebraic set.

Theorem

Let A be a semialgebraic subset of K^n. If a point $a \in K^n$ lies in the closure (in the K-topology) of $A \setminus \{a\}$, then there is a semialgebraic map $\varphi : R \longrightarrow K^n$ given by algebraic power series such that

$$\varphi(0) = a \quad \textit{and} \quad \varphi(R \setminus \{0\}) \subset A \setminus \{a\}.$$

Theorem

Let A be an $\mathcal{L}$-definable subset of K^n. If a point $a \in K^n$ lies in the closure (in the K-topology) of $A \setminus \{a\}$, then there exist a semialgebraic map $\varphi : R \longrightarrow K^n$ given by algebraic power series and an $\mathcal{L}$-definable subset E of R with accumulation point 0 such that

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Theorem

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$$\varphi(0) = a \quad \text{and} \quad \varphi(E \setminus \{0\}) \subset A \setminus \{a\}.$$

The proofs rely on resolution of singularities and the closedness theorem.

Piecewise continuity of definable functions

Theorem

Let $A \subset K^n$ and $f : A \longrightarrow \mathbb{P}^1(K)$ be an $\mathcal{L}^P$-definable function in the three-sorted language of Denef–Pas. Then f is piecewise continuous, i.e. there is a finite partition of A into $\mathcal{L}^P$-definable locally closed subsets $A_1, \ldots, A_s$ of K^n such that the restriction of f to each A_i is continuous.

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The proof makes use of the closedness theorem and some basic properties of dimension.

The Tietze–Urysohn extension theorem

Theorem

Every continuous $\mathcal{L}$-definable function $f : A \longrightarrow K$ on a closed subset A of K^n has a continuous $\mathcal{L}$-definable extension F to K^n.

The Tietze–Urysohn extension theorem

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Every continuous $\mathcal{L}$-definable function $f : A \longrightarrow K$ on a closed subset A of K^n has a continuous $\mathcal{L}$-definable extension F to K^n.

The proof relies on resolution of singularities and on various consequences of the closedness theorem.

An analytic structure is determined by a certain (separated or strictly convergent) Weierstrass system $\mathcal{A}$ defined on a commutative ring A with unit. It is described in a two sorted, analytic language $\mathcal{L}$, explained later. The theory of valued fields with analytic structure, developed by Cluckers–Lipshitz–Robinson [CLR, CL1, CL2], has a long history. It unifies earlier work of many mathematicians, as Denef, van den Dries, Haskell, Lipshitz, Macintyre, Macpherson, Marker and Robinson.

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It is enough to prove the closedness theorem in the separated case, because every strictly convergent analytic structure can be extended in a definitional way (by Henselian functions) to a separated one Examples of the former are the classical, complete rank one valued fields with the Tate algebra of strictly convergent power series.

Separated analytic structure

Recall the concept of a separated analytic structure. Let A be a commutative ring with unit and with a fixed proper ideal $I \subsetneq A$. A *separated* (A, I)-*system* is a certain system $\mathcal{A}$ of A-subalgebras $A_{m,n} \subset A[[\xi, \rho]]$, $m, n \in \mathbb{N}$; here $A_{0,0} = A$. Two kinds of variables, ξ and ρ, play different roles. Roughly speaking, the variables ξ vary over the valuation ring (or the closed unit disc) K°, and the variables ρ vary over the maximal ideal (or the open unit disc) $K^{\circ\circ}$ and are used to witness strict inequalities. $\mathcal{A}$ is called a *separated pre-Weierstrass system* if two usual Weierstrass division theorems hold in each $A_{m,n}$.

A pre-Weierstrass system $\mathcal{A}$ satisfying a condition on the so-called rings of $\mathcal{A}$-fractions is called a *separated Weierstrass system*. That condition may be regarded as a kind of weak Noetherian property.

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A *separated analytic $\mathcal{A}$-structure* on a valued field K is a collection of homomorphisms $\sigma_{m,n}$ from $A_{m,n}$ to the ring of K°-valued functions on $(K^{\circ})^m \times (K^{\circ\circ})^n$, $m, n \in \mathbb{N}$, such that
1) $\sigma_{0,0}(I) \subset K^{\circ\circ}$;
2) $\sigma_{m,n}(\xi_i)$ and $\sigma_{m,n}(\rho_j)$ are the i-th and $(m+j)$-th coordinate functions on $(K^{\circ})^m \times (K^{\circ\circ})^n$, respectively;
3) $\sigma_{m+1,n}$ and $\sigma_{m,n+1}$ extend $\sigma_{m,n}$, where functions on $(K^{\circ})^m \times (K^{\circ\circ})^n$ are identified with those functions on

$$(K^{\circ})^{m+1} \times (K^{\circ\circ})^n \quad \text{or} \quad (K^{\circ})^m \times (K^{\circ\circ})^{n+1}$$

which do not depend on the coordinate ξ_{m+1} or ρ_{n+1}, respectively.

Analytic $\mathcal{A}$-structures preserve composition. If the ground field K is non-trivially valued, then the function induced by a power series from $A_{m,n}$, $m, n \in \mathbb{N}$, is the zero function iff the image in K of each of its coefficients is zero.

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When considering a particular field K with analytic $\mathcal{A}$-structure, one may assume that $\ker \sigma_{0,0} = (0)$. Indeed, replacing A by $A/\ker \sigma_{0,0}$ yields an equivalent analytic structure on K with this property. Then $A = A_{0,0}$ can be regarded as a subring of K°. Moreover, by extension of parameters, one can get a (unique) separated Weierstrass system $\mathcal{A}(K)$ over $(K^\circ, K^{\circ\circ})$ and K has separated analytic $\mathcal{A}(K)$-structure.

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A separated analytic $\mathcal{A}$-structure on K can be uniquely extended to any algebraic extension K' of K. The above properties remain valid for strictly convergent analytic structures. Finally, every valued field with separated analytic structure is Henselian.

Analytic language $\mathcal{L}$

We begin by defining the semialgebraic language $\mathcal{L}_{Hen}$. It is a two sorted language with the main, valued field sort K, and the auxiliary RV-sort

$$RV = RV(K) := RV^* \cup \{0\}, \quad RV^*(K) := K^\times/(1 + K^{\circ\circ});$$

here $A^\times$ denotes the set of units of a ring A. The language of the valued field sort is the language of rings $(0, 1, +, -, \cdot)$. The language of the auxiliary sort is the so-called inclusion language. The only map connecting the sorts is the canonical map

$$rv : K \to RV(K), \quad 0 \mapsto 0.$$

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We have the canonical exact sequence

$$1 \to \widetilde{K}^\times \to RV(K)^* \to \Gamma \to 0,$$

which splits iff the valued field K has an angular component map.

The analytic language $\mathcal{L} = \mathcal{L}_{Hen,\mathcal{A}}$ is the semialgebraic language $\mathcal{L}_{Hen}$ augmented on the valued field sort K by the function $1/x$ (with $1/0 := 0$) and the names of all functions of the system $\mathcal{A}$, together with the induced language on the auxiliary sort RV. A power series $f \in A_{m,n}$ is construed via the analytic $\mathcal{A}$-structure on their natural domains (and as zero outside them):

$$\sigma(f) : (K^{\circ})^m \times (K^{\circ\circ})^n \to K^{\circ}.$$

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In the equicharacteristic case, however, the induced language on the auxiliary sort RV coincides with the semialgebraic inclusion language. Therefore the residue field $\widetilde{K}$ is orthogonal to the value group Γ_K, i.e. every definable set in $\widetilde{K}^m \times \Gamma_K^n$ is a finite union of the Cartesian products of some sets definable in $\widetilde{K}^m$ (in the language of rings) and in Γ_K^n (in the language of ordered groups). The orthogonality property is often used both in the algebraic and analytic case treated in our papers [N2, N3, N4].

Let $\mathcal{T}_{Hen,\mathcal{A}}$ be the theory of all Henselian valued fields of characteristic zero with separated analytic $\mathcal{A}$-structure. Denote by $\mathcal{L}^*$ the analytic language $\mathcal{L}$ augmented by all Henselian functions

$$h_m : K^{m+1} \times RV(K) \to K, \quad m \in \mathbb{N},$$

defined by means of a version of Hensel's lemma. We need the two theorems on b-minimality and term structure from [CL1]):

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defined by means of a version of Hensel's lemma. We need the two theorems on b-minimality and term structure from [CL1]):

Theorem

1) The theory $\mathcal{T}_{Hen,\mathcal{A}}$ eliminates valued field quantifiers, is b-minimal with centers and preserves all balls. In particular, it admits b-minimal cell decomposition.
2) Let $f : X \to K$, $X \subset K^n$, be an $\mathcal{L}(B)$-definable function for some set of parameters B. Then there exist an $\mathcal{L}(B)$-definable function $g : X \to S$ with S auxiliary and an $\mathcal{L}^*(B)$-term t such that

$$f(x) = t(x, g(x)) \quad \textit{for all} \;\; x \in X.$$

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