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ON THE MAXIMUM OF THE JAKUBOWSKI FUNCTIONAL IN THE CLASS OF HOLOMORPHIC AND UNIVALENT FUNCTIONS

W. Majchrzak and A. Szwankowski (Łódź)

1. Introduction

Let S stand for the well-known class of functions

(1) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

holomorphic and univalent in the disc $\Delta = \{z : |z| < 1\}$.

The present report concerns the maximum of the functional

(2) $$H(f) = |a_2^m(a_3 - \alpha a_2^2)|, \qquad \alpha \in \mathbb{R}, \quad m = 1, 2, \dots$$

defined in the class S.

As is well known, the factor $|a_2^m|$, $m = 1, 2, \dots$, is maximized by the Koebe functions, while the other factor $|a_3 - \alpha a_2^2|$, with $\alpha \in (0, 1)$, attains its maximum for functions which are not Koebe ones. So, the question is whether there exist $\alpha \in (0,1)$ with which the extremal functions for (2) are the Koebe functions.

In the case $m = 2$ ([3]) it was proved that, for $\alpha \leq 3/8$ and $\alpha \geq 7/8$, this really holds. Moreover, in [4], [5], the extrema of the functional $a_2^m(a_3 - \alpha a_2^2)$, $\alpha \in \mathbb{R}$, $m = 1, 2, \ldots$, for functions from S_R were found, where S_R is the subclass of S of all functions with real coefficients a_n, $n = 2, 3, \ldots$.

Functional (2) was also studied in the linearly invariant classes ([1], [2]) where it was called the Jakubowski functional (see also [6]).

2. Differential-functional equation for extremal functions

Let us consider the functional

$$H^*(f) = \mathrm{Re}\{a_2^m(a_3 - \alpha a_2^2)\} \tag{3}$$

defined in the class S, where α is real and $m = 1, 2, \ldots$.

Note first that, together with the function $f(z)$, also the function $e^{-i\Theta} f(e^{i\Theta} z)$, $\Theta \in \mathbb{R}$, belongs to the class S. In consequence, the maximum of the functional (2) is identical in this class with that of functional (3). We shall therefore confine ourselves to the latter. Besides, as we have already observed, it is enough to carry out the proof for $\alpha \in (0,1)$.

By the compactness of S and the continuity of functional (3) the maximum of $H^*(f)$ in S exists for arbitrarily fixed parameters α and m.

Any function $f_{\alpha,m} \in S$, such that

$$H^*(f_{\alpha,m}) = \max_{f \in S} H^*(f),$$

will be called extremal.

Since for any extremal function $\operatorname{grad} H^*(f) \neq 0$ we can use the well-known Schaeffer-Spencer theorem ([7]) which assures that any extremal function $f = f_{\alpha,m}$ satisfies the following equation:

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{lf(z) + k}{f^2(z)} = \frac{\bar{k}z^4 + \bar{l}z^3 + B_0 z^2 + lz + k}{z^2}, \qquad z \in \Delta, \tag{4}$$

where

$$\begin{aligned} B_0 &= (m+2)a_2^m(a_3 - \alpha a_2^2),\\ l &= a_2^{m-1}[m(a_3 - \alpha a_2^2) + 2(1-\alpha)a_2^2],\\ k &= a_2^m. \end{aligned}$$

Here we have (see [7]) that $B_0 > 0$ and the right-hand side of (4) is nonnegative on $|z| = 1$ with at least one double zero z_0 on this circle.

It can be proved that only the following forms of (4) are possible:

(a) $$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{lf(z)+k}{f^2(z)} = \bar{k}\frac{(z-z_0)^2(z-z_1)(z-z_2)}{z^2}, \qquad l \neq 0,$$

or

(b) $$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{lf(z)+k}{f^2(z)} = \bar{k}\frac{(z-z_0)^2(z-z_3)^2}{z^2}, \qquad l \neq 0,$$

or

(c) $$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{lf(z)+k}{f^2(z)} = \bar{k}\frac{(z-z_0)^4}{z^2}, \qquad l \neq 0,$$

or

(d) $$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{k}{f^2(z)} = \bar{k}\frac{(z-z_0)^2(z-z_3)^2}{z^2}, \qquad l = 0,$$

where $z_0 = e^{i\psi}$, $z_1 = \rho e^{i\varphi}$, $z_2 = 1/\bar{z}_1$, $0 < \rho < 1$, $\psi \in \langle 0, \pi/2 \rangle$, φ is real, $|z_3| = 1$, $z_3 \neq z_0$.

In cases (a), (b), (c), (d) mentioned above we obtain the following four lemmas, respectively:

Lemma 1. *If, for $\alpha \in (0, 3m/(4m+8)) \cup ((3m+8)/(4m+8), 1)$, an extremal function satisfies equation (a), then it is a Koebe function of the form*

$$f(z) = z/(1+\bar{z}_0 z)^2,$$

and

(5) $$\max_{f\in S} H^*(f) = \begin{cases} 2^m(3-4\alpha) & \textit{for} \quad 0 < \alpha < 3m/(4m+8) \\ 2^m(4\alpha-3) & \textit{for} \quad (3m+8)/(4m+8) < \alpha < 1. \end{cases}$$

Moreover, for $\alpha \in \langle 3m/(4m+8), (3m+8)/(4m+8) \rangle$, the extremal function does not satisfy equation (a).

Lemma 2. *If an extremal function satisfies equation (b), then the maximum of functional (3) is expressed alternatively:*

(6) $$\max_{f\in S} H^*(f) = \frac{2m^{m/2}}{(1-\alpha)^{m/2}(m+2)^{m/2+1}} \left(2 + \exp\frac{2\alpha-1}{1-\alpha}\right)^{m/2+1} \exp\left[\frac{(1-2\alpha)(m+2)}{2(1-\alpha)}\right]$$

only for α *and* m *which satisfy the condition:*

$$\frac{1}{4(1-\alpha)}\left(1+2\exp\frac{1-2\alpha}{1-\alpha}\right) < \frac{m+2}{m} < (1-\alpha)\left(2+\exp\frac{2\alpha-1}{1-\alpha}\right), \tag{7}$$

or

$$\max_{f\in S} H^*(f) = \frac{2^{m+1}}{m+2}\cos^m\varphi(1-\log\cos\varphi)^m(1+2\cos^2\varphi) \tag{8}$$

where φ *is the function inverse to the function*

$$\alpha = 1 + \frac{\frac{m}{m+2}(1+2\cos^2\varphi) - 4\cos^2\varphi(1-\log\cos\varphi)}{4\cos^2\varphi(1-\log\cos\varphi)^2}, \qquad \varphi \in \left(0, \frac{\pi}{2}\right). \tag{9}$$

Moreover, inequalities (7) are not satisfied if $\alpha \geq 3m/(4m+8)$ and relation (9) between α and φ implies $\alpha > 3m/(4m+8)$.

Lemma 3. *If an extremal function satisfies equation (c), then*

$$\max_{f\in S} H^*(f) = 2^m(3-4\alpha) \tag{10}$$

but it is valid for $\alpha = 3m/(4m+8)$ *only.*

Lemma 4. *If an extremal function satisfies equation (d), then*

$$\max_{f\in S} H^*(f) = \frac{2m^{m/2}}{(1-\alpha)^{m/2}(m+2)}, \tag{11}$$

where α *and* m *must satisfy the inequality*

$$0 < \alpha \leq \frac{3m+8}{4m+8}. \tag{12}$$

3 The main theorem

Making use of Lemmas 1–4, we obtain

Theorem. *For any function* $f \in S$ *and* $m = 1, 2, \ldots,$ *the following estimate is true:*

$$H(f) \leq \begin{cases} 2^m(3-4\alpha) & \textit{for } \alpha \leq \dfrac{3m}{4m+8}, \\ \frac{2^{m+1}}{m+2}\cos^m\varphi(1-\log\cos\varphi)^m(1+2\cos^2\varphi) & \textit{for } \dfrac{3m}{4m+8} \leq \alpha \leq \alpha_0, \\ \frac{2}{m+2}\left[\frac{m}{(1-\alpha)(m+2)}\right]^{m/2} & \textit{for } \alpha_0 \leq \alpha \leq \dfrac{3m+8}{4m+8}, \\ 2^m(4\alpha-3) & \textit{for } \alpha \geq \dfrac{3m+8}{4m+8}, \end{cases}$$

where φ *is the function inverse to the function* $\alpha = \alpha(\varphi)$ *defined by*

$$\alpha = 1 + \frac{\frac{m}{m+2}(1+2\cos^2\varphi) - 4\cos^2\varphi(1-\log\cos\varphi)}{4\cos^2\varphi(1-\log\cos\varphi)^2}, \qquad \varphi \in \langle 0, \varphi_0),$$

while φ_0 *is the smallest positive root of the equation*

$$\left[4\cos^2\varphi(1-\log\cos\varphi) - \frac{m}{m+2}(1+2\cos^2\varphi)\right](1+2\cos^2\varphi)^{2/m} = \frac{m}{m+2},$$

and $\alpha_0 = \alpha(\varphi_0)$. *The estimate is exact.*

A proof of estimate consists in comparison of all values of functional (2) founded in Lemmas 1–4 for extremal functions taking into account the conditions which confine the range of parameters α and m at any case considered.

The detailed proofs of all theorems cited above will be published in Rev. Roumanie Math. Pures et Appl. in 1994.

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O maksimum funkcjonału Jakubowskiego w klasie funkcji holomorficznych i jednolistnych

Streszczenie. W pracy wyznacza się maksimum funkcjonału $H(f) = |a_2^m(a_3 - \alpha a_2^2)|$, $m = 1, 2, \ldots$, $\alpha \in \mathbb{R}$, określonego w klasie S funkcji f holomorficznych i jednolistnych w kole jednostkowym postaci $f(z) = z + \sum_{n=2}^\infty a_n z^n$, $|z| < 1$.

Metoda dowodu oszacowania polega na wykorzystaniu równania różniczkowo-funkcyjnego dla funkcji klasy S ekstremalnych względem funkcjonałów zależnych od skończonej ilości współczynników a_n tych funkcji (p. [7]).

Otrzymane oszacowanie jest dokładne.

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