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THE PROBLEM OF CONVEXITY AND COMPACTNESS OF SOME CLASSES OF CARATHÉODORY FUNCTIONS

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1. The article belongs to the cycle of papers [1]–[5], where different classes of functions defined by conditions on the unit circle $\mathbb{T}$ were studied. The results from [6] are completed. As usual, we shall denote by $\mathbb{C}$ the complex plane, by $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$ the unit disc, by $\mathbb{T} = \{z \in \mathbb{C};\ |z| = 1\}$ the unit circle.

Let P denote the class of functions of the form

$$p(z) = 1 + q_1 z + \ldots + q_n z^n + \ldots \tag{1}$$

holomorphic in $\mathbb{D}$ with $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$ and, for a given set $F \subset \mathbb{C}$, let $F_\tau = \{\xi \in \mathbb{C};\ e^{-i\tau}\xi \in F\}$ be the set arising by rotation of F through the angle τ.

Definition 1 (see [2], [3], [4]). Let $0 \le b < 1$, $b < B$, $0 < \alpha < 1$ be fixed real numbers.

a) Let $F \subset \mathbb{T}$ be a closed set of Lebesgue measure $2\pi\alpha$. By $P(B, b, \alpha; F)$ we denote the class of functions $p \in P$ satisfying the following conditions: there exists $\tau = \tau(p) \in \langle -\pi, \pi\,)$ such that

$$\operatorname{Re} p(e^{i\theta}) \ge B \quad \text{a. e. on } F_\tau$$

and

$$\operatorname{Re} p(e^{i\theta}) \ge b \quad \text{a. e. on } \mathbb{T}\backslash F_\tau.$$

b) By $P(B,b,\alpha)$ we denote the class of functions $p\in P$ such that there exists a closed set $F=F(p)$, $F\subset\mathbb{T}$, of Lebesgue measure $2\pi\alpha$ such that

$$\operatorname{Re} p(e^{i\theta})\geq B \quad \text{a. e. on } F \tag{2}$$

and

$$\operatorname{Re} p(e^{i\theta})\geq b \quad \text{a. e. on } \mathbb{T}\backslash F. \tag{3}$$

c) For a fixed $\tau\in\langle-\pi,\pi)$, by $P(B,b,\alpha;F,\tau)$ we denote the set of all functions from $P(B,b,\alpha;F)$ satisfying (2) and (3) on F_τ and $\mathbb{T}\backslash F_\tau$, respectively.

d) By $\tilde{P}(B,b,\alpha)$ we denote class of functions $p\in P$ such that there exists an open arc $I=I(p)\subset\mathbb{T}$ of Lebesgue measure $2\pi\alpha$ such that (2) and (3) are fulfilled for $F=\bar{I}$.

e) Let $F\subset\mathbb{T}$ be a fixed closed set of Lebesgue measure $2\pi\alpha$. By $\check{P}(B,b,\alpha;F)$ we denote the class of functions $p\in P$ fulfilling (2) and (3).

In paper [2] (Th. 3; see also [4], L.1) it was proved that the class $\tilde{P}(B,b,\alpha)$ is compact in the topology given by the uniform convergence on compact subsets of $\mathbb{D}$, but it is not convex (Th. 5). On the other hand, each class $\check{P}(B,b,\alpha;F)$, especially $\check{P}(B,b,\alpha;\bar{I})$ (see e.g. [3], Th. 6 and [4] L.1) is convex. The class $P(B,b,\alpha;F,\tau)$ defined in Def. 1 (c) are convex and compact ([4], part 3) and the classes $P(B,b,\alpha;F)$ are also compact ([4], part 3). In this paper we shall discuss the problem of convexity and connectedness for the classes $P(B,b,\alpha;F)$ and the problem of convexity and compactness for the class $P(B,b,\alpha)$.

2. In the sequel, we denote by $l(A)$ the normalized Lebesgue measure on $\mathbb{T}$ ($l(\mathbb{T})=1$). We shall need the following

Lemma 1. *Let $F\subset\mathbb{T}$ be a closed set, $l(F)=\alpha$, $0<\alpha<1$. Then for each $\tau\in\langle-\pi,\pi)$, there exists $\delta>0$ such that $l(F_{\tau+h}\cap F_\tau)<l(F_\tau)$ for each h, $0<|h|<\delta$.*

Proof. Without loss of generality we can choose $\tau_0=0$ and F_0 to be perfect (because the set of isolated points of F_0 is countable and hence a set of Lebesgue measure zero). Denote by $\mathbb{D}_{F_0}$ the set of density points of F_0 (i.e. $\xi\in\mathbb{D}_{F_0}$ if and only if $\lim_{r\to0}\frac{l(F_0\cap B(\xi,r))}{2r}=1$ where $B(\xi,r)$ is the arc $\mathbb{T}$ with centre at the point ξ and $l(B(\xi,r))=2r$. Then (see [7], Exercise 11, p. 177) $l(\mathbb{D}_{F_0})=l(F_0)=\alpha>0$. Hence each interval containing a point of F_0 contains a point of $\mathbb{D}_{F_0}$. Denote $G_0=\mathbb{T}\backslash F_0$, so $l(G_0)=1-l(F_0)=1-\alpha>0$. G_0 is an open subset of $\mathbb{T}$, hence G_0 is the sum of a nonvoid finite or countable family of mutually disjoint open arcs $G_i\subset\mathbb{T}$. Let $l(G_{i_0})\geq l(G_i)$ for every i and put $\delta=l(G_{i_0})$. The endpoints ξ_0, ξ_1 of G_{i_0} are lying in F_0. So, by rotating F_0 through any angle h, $|h|<\delta$, $G_{i_0}\cap F_0$ contains ξ_0 and ξ_1, and so, in any case, a point $\xi\in\mathbb{D}_{F_0}$ and an arc $B(\xi,r_0)$. Take $0<r<r_0$ such that $l(F_0\cap B(\xi,r))>\frac{1}{2}l(B(\xi,r))=r$. Then $F_0\cap F_h\subset F_0\backslash(F_0\cap B(\xi,r))$, so $l(F_0\cap F_h)\leq l(F_0)-r<l(F_0)$.

Theorem 1. *$P(B,b,\alpha;F)$ is not convex.*

Proof. By Theorem 2 of [6] (this volume, p. 17), there exists a non-constant function $p\in P(B,b,\alpha;F)$ such that $\operatorname{Re} p(e^{i\Theta})=B$ a.e. on F, $\operatorname{Re} p(e^{i\Theta})=b$

on $\mathbb{T}\backslash F$ (cf. [4] , (12) and [6], Th. 2). Take $0 < \tau < \min(\alpha, 1-\alpha)$ and define $p_\tau(z) = p^{-2\pi i\tau}z)$, $z \in \mathbb{D}$. Obviously, $p_\tau \in P(B, b, \alpha; F, \tau)$. Join p, p_τ by the segment $p_\lambda = \lambda p_\tau + (1-\lambda)p$, $0 \le \lambda \le 1$. Clearly, $p_\lambda(0) = 1$. One has $\operatorname{Re} p_\lambda(\xi) \le \lambda b + (1-\lambda)B < B$ on $\mathbb{T}\backslash F_\tau$ for $\lambda > 0$ and $\operatorname{Re} p_\lambda(\xi) \le \lambda B + (1-\lambda)b < B$ on $\mathbb{T}\backslash F$ for $\lambda < 1$. So, only a.e. on $F_\tau \cap F$ $\operatorname{Re} p_\lambda(\xi) \ge B$ can be fulfilled. By Lemma 1, there exists some $\delta = \delta(F) > 0$ such that $l(F \cap F_h) < l(F)$ for each h, $|h| < \delta$. Hence, for each $\lambda \in (0,1)$, p_λ does not belong to $P(B, b, \alpha; F)$.

Theorem 2. *$P(B, b, \alpha; F)$ is arcwise connected (and thus connected).*

Proof. Let $p_1, p_2 \in P(B, b, \alpha; F)$. Then there exists $\tau_1, \tau_2 \in \langle -\pi, \pi)$ such that $p_k \in P(B, b, \alpha; F, \tau_k)$, $k = 1, 2,$. Since the classes $P(B, b, \alpha; F, \tau)$ are convex, we can join p_1, p_2 by a segment with $p_{F_{\tau_1}} + 1 - \eta$, $p_{F_{\tau_2}} + 1 - \eta$, respectively, and then $p_{F_{\tau_1}} + 1 - \eta$ with $p_{F_{\tau_2}} + 1 - \eta$ by the arc $\tau \to p_{F_\tau} + 1 - \eta$, $\tau_1 \le \tau \le \tau_2$ (cf. [4], Remark 2).

Remark 1. In the case $B \le 1$, the assertion of Theorem 2 is obvious because $p(z) \equiv 1$, $z \in \mathbb{D}$, belongs to $P(B, b, \alpha; F, \tau)$ for each $\tau \in \langle -\pi, \pi)$.

Remark 2. All the properties of the class $P(B, b, \alpha; F)$ which we have examined up to now (i.e. compactness, convexity and connectedness) require non-trivial means from real analysis for their proofs, but can be proved almost trivially if we restrict our attention to the classes $P(B, b, \alpha; F, \tau)$. In this context, the following properties can be of some interest.

Lemma 2. *For each $\tau \in \langle -\pi, \pi)$, we have*

$$\lim_{h\to 0} l(F_{\tau+h} \cap F_\tau) = l(F_\tau).$$

Proof. We can suppose $\tau = 0$ and write $F_\tau = F_0$. Since $\chi_{F_h \cap F_0} = \chi_{F_h}\chi_{F_0}$, we have

$$\begin{aligned} l(F_0) - l(F_f \cap F_0) &= \int_{-\pi}^{\pi} (\chi_{F_0} - \chi_{F_0}\chi_{F_h}) \frac{dt}{2\pi} = \int_{-\pi}^{\pi} (\chi_{F_0}^2 - \chi_{F_0}\chi_{F_h}) \frac{dt}{2\pi} \\ &= \int_{-\pi}^{\pi} \chi_{F_0}(\chi_{F_0} - \chi_{F_h}) \frac{dt}{2\pi} \le \int_{-\pi}^{\pi} |\chi_{F_0} - \chi_{F_h}| \frac{dt}{2\pi} \\ &= \int_{-\pi}^{\pi} |\psi_{F_0}(t+h) - \psi_{F_0}(t)| \frac{dt}{2\pi} \end{aligned}$$

where we denoted $\psi_{F_0}(t) = \chi_{F_0}(e^{it})$. But $\lim_{h\to 0} \int_{-\pi}^{\pi} |\psi_{F_0}(t+h) - \psi_{F_0}(t)| dt = 0$ (see e.g. [7], Th. 9.5, p. 183) and $\lim_{h\to 0} l(F_0 \cap F_h) = l(F_0)$.

Theorem 3. *Let $\eta < 1$. Then there exists $\tau_i = \tau_i(F)$, $i = 1, 2$, such that, for each $\tau \in (-\tau_i, \tau_i)$, $i = 1, 2$, we have*

(i) *$P(B, b, \alpha; F, \tau) \ne P(B, b, \alpha; F, 0)$ for $0 < |\tau| < \tau_1$,*
(ii) *$P(B, b, \alpha; F, \tau) \cap P(B, b, \alpha; F, 0) \ne \emptyset$ for $|\tau| < \tau_2$.*

Proof. (i) By Lemma 1, there exists $\tau_1 > 0$ such that, for each $\tau \in (-\tau_1, \tau_1)$, one has $l(F \cap F_\tau) < l(F)$. Hence the function $\tilde{P}_F(z) = b + (B-b)h(z; F) + (1-\eta)\frac{e^{i\gamma}+z}{e^{i\gamma}-z}$, γ real, $z \in \mathbb{D}$, does not belong to $P(B, b, \alpha; F, \tau)$ since

$\mathrm{Re}\,\tilde{P}_F(e^{i\Theta}) = b < B$ a.e. on $F_\tau \backslash F$, $l(F_\tau \backslash F) = l(F_\tau) - l(F_\tau \cap F) = l(F) - l(F \cap F_\tau) > 0$.

(ii) Define

$$f(e^{it}) = B \qquad \text{a.e. on } \ F \cup F_\tau,$$
$$f(e^{it}) = b \qquad \text{on } \ \mathbb{T}\backslash(F \cup F_\tau)$$

and define

$$p(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\frac{e^{it}+z}{e^{it}-z}dt, \qquad z \in \mathbb{D}.$$

Then $\mathrm{Re}\,p(e^{i\Theta}) = B$ a.e. on $F \cup F_\tau$, $\mathrm{Re}\,p(e^{i\Theta}) = b$ on $\mathbb{T}\backslash(F \cup F_\tau)$. It is clear that $\mathrm{Re}\,p$ fulfils condition (2) a.e.on F and F_τ and condition (3) on $\mathbb{T}\backslash F$ and $\mathbb{T}\backslash F_\tau$. An easy calculation gives

$$p(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})dt = \eta + (B-b)[l(F) - l(F \cap F_\tau)]. \tag{4}$$

But, by Lemma 2, $\lim_{\tau\to 0} l(F \cap F_\tau) = l(F)$. Hence, by (4) and on account of $\eta < 1$, there exists $\tau_2 > 2$ such that $p(0) < 1$ for $|\tau| < \tau_2$ Then the function

$$\tilde{p}(z) = p(z) + (1 - p(0))\frac{e^{i\gamma}+z}{e^{i\gamma}-z}, \qquad \gamma\text{- real}, \ \ z \in \mathbb{D},$$

belongs to $P(B, b, \alpha; F, \tau) \cap P(B, b, \alpha; F, 0)$ for each $\tau \in (-\tau_2, \tau_2)$.

3. Now, we shall consider the problem of compactness and convexity for the class $P(B, b, \alpha)$.

The estimates of the linear functionals $\mathrm{Re}\,p(z)$ and $\mathrm{Im}\,p(z)$, $z \in \mathbb{D}$ fixed, given in [5] (Th. 2), and also the estimates (see [5], Remark 3) of the convex functionals $|q_k|$, $k = 1, 2, \ldots$, are interesting from the following point of view: they are valid on all the closed convex hull of $P(B, b, \alpha)$, although $P(B, b, \alpha)$ is neither convex nor compact (this wil be shown in this section). Recall, that the topology on $P(B, b, \alpha)$ is the restriction of the topology given by uniform convergence on compact subsets of $\mathbb{D}$ on the set of all functions holomorphic in $\mathbb{D}$ and that the class P is compact and hence $P(B, b, \alpha)$ is relatively compact in P in this topology.

Theorem 4. *The class $P(B, b, \alpha)$ is neither convex nor compact.*

Proof. First, we prove that $P(B, b, \alpha)$ is not convex. Let $p_F(z) = b + (B - b)h(z; F)$, $z \in \mathbb{D}$ (see, for example, [6], (14)). Take $p_1(z) = p_{F_1}(z) + (1-\eta)\frac{1+z}{1-z}$, $p_2(z) = p_{F_2} + (1-\eta)\frac{1+z}{1-z}$, $z \in \mathbb{D}$, where the closed sets F_i, $i = 1, 2$, are chosen in such a manner, that $0 \le m(F_1 \cap F_2) < \alpha$. Put $p_t = tp_1 + (1-t)p_2$, $0 < t < 1$. Since $Re\frac{1+z}{1-z} = 0$ a.e. on $\mathbb{T}$, $\mathrm{Re}\,p_{F_i} = B$ a.e. on F_i, $\mathrm{Re}\,p_{F_i} = b$ a.e. on $\mathbb{T}\backslash F_i$ and $tb + (1-t)B < B$ for $0 < t < 1$, $\mathrm{Re}\,p_t = B$ a.e. on $F_1 \cap F_2$ and $\mathrm{Re}\,p_t < B$ a.e. on $\mathbb{T}\backslash F_1 \cap F_2$. Since $m(F_1 \cap F_2) < \alpha$, p_t does not satisfy (2) and so does not belong to $P(B, b, \alpha)$.

Now, we prove, that $P(B, b, \alpha)$ is not compact. Since $P(B, b, \alpha) \subset P$, it is sufficient to prove that $P(B, b, \alpha)$ is not closed. Put

$$p_n(z) = b + (B-b)h_{F_n}(z) + (1-\eta)\frac{1+z}{1-z}, \quad z \in \mathbb{D}, \tag{5}$$

where

$$F_n = \bigcup_{k=1}^{n} F_n^k, \qquad F_n^k = \left\{ z \in \mathbb{T};\ z = e^{\frac{2k\pi i}{n}} e^{i\rho},\ -\frac{\alpha\pi}{n} \le \rho \le \frac{\alpha\pi}{n} \right\},$$

and

$$h_{F_n}(z) = \alpha + 2\sum_{r=1}^{\infty} \frac{\sin \alpha\pi r}{\pi r} z^{rn}, \qquad z \in \mathbb{D}$$

(see [3], p. 2). For $z \in \mathbb{D}$, $|z| \le \rho < 1$, we have

$$|h_{F_n}(z) - \alpha| \le 2\sum_{r=1}^{\infty} \frac{|\sin \alpha\pi r|}{\pi r} \rho^{rn} \le 2\rho^n \sum_{r=0}^{\infty} (\rho^n)^r = \frac{2\rho^n}{1-\rho^n},$$

and so, the sequence $\{h_{F_n}\}_{n=1}^{\infty}$ is uniformly convergent to the constant function α on every compact subset of $\mathbb{D}$. Denoting $p_0(z) = \eta + (1-\eta)\frac{1+z}{1-z}$, $\eta = \alpha B + (1-\alpha)b$, and using (5) we see that $p_n(z) \to p_0(z)$ uniformly on compact subsets of $\mathbb{D}$. But the function $Re p_0$ is equal η a.e. on $\mathbb{T}$, since $Re\frac{1+z}{1-z}$ is zero a.e. on $\mathbb{T}$. Since $\eta < \alpha B + (1-\alpha)B = B$, p_0 does not fulfil (2), and so, does not belong to $P(B, b, \alpha)$.

Remark 3. The idea of the sequence $\{p_n\}$ comes from Theorem 5 of [6]: the function p_n realizes the maximum modules of the n-th coefficient in the class $P(B, b, \alpha)$. The measure μ_n in the Poisson representation of p_n is the sum of two parts: the (absolutely continuous) part $[b + (B-b)\chi_{F_n}(t)]\frac{dt}{2\pi}$ and the (singular) part $(1-\eta)\varepsilon_0$ where ε_0 is the Dirac measure sitting a the point $t = 0$. Now, intuitively, the measures $\chi_{F_n}(t)\frac{dt}{2\pi}$ spread to the measure $\alpha\frac{dt}{2\pi}$ and the limit function $p_0(z)$ which is represented by the limit measure $\eta\frac{dt}{2\pi} + (1-\eta)\varepsilon_0$ and does not belong to $P(B, b, \alpha)$.

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Zagadnienia wypukłości i zwartości pewnych klas funkcji Carathéodory'ego

Streszczenie. Niech P oznacza znaną klasę funkcji $p(z) = 1 + q_1 z + \dots$ holomorficznych w kole jednostkowym $\mathbb{D}$ i takich, że $\operatorname{Re} p(z) > 0$ w $\mathbb{D}$. W artykule są badane zagadnienia wypukłości lub zwartości podklas $P(B, b, \alpha; F)$ i $P(B, b, \alpha)$ rodziny P określonych w Definicji 1. Praca należy do cyklu publikacji [1]–[5], gdzie były rozważane różne klasy funkcji holomorficznych w $\mathbb{D}$ i spełniających na okręgu jednostkowym $\mathbb{T}$ pewne warunki. Stanowi uzupełnienie rezultatów z pozycji [6].

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