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ON COEFFICIENT ESTIMATES IN A CLASS OF CARATHÉODORY FUNCTIONS WITH POSITIVE REAL PART

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1. This article belongs to the cycle of papers ([4], [5], [6]) where different classes of functions defined by conditions on the unit circle were studied. The results of papers [5], [6] are generalized. Omitted proofs and other new results will be published in [7]. As usual, we shall denote by $\mathbb{C}$ the complex plane, by $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$ the unit disc, by $\mathbb{T} = \{z \in \mathbb{C};\ |z| = 1\}$ the unit circle. In our further considerations, we shall treat $\mathbb{T}$ as the subset of $\mathbb{C}$ with the induced topology on the one hand, on the other hand, as a set homeomorphic to $\mathbb{T}$, namely, as the subset $\langle -\pi, \pi)$ of the real line $\mathbb{R}$, endowed with the factor topology $\mathbb{R}/2\pi\mathbb{Z}$ where $\mathbb{Z}$ is the set of integers. Therefore we shall sometimes treat the function $f(e^{i\Theta}) : \mathbb{T} \to \mathbb{C}$ as a function $f(t) : \langle -\pi, \pi) \to \mathbb{C}$.

Let P denote the class of functions of the form

$$p(z) = 1 + q_1 z + \ldots + q_n z^n + \ldots \tag{1}$$

holomorphic in the unit disc $\mathbb{D}$ with $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$ ([2]).

Let us recall some properties of real parts of functions from P, which will be essential in what follows:

(a) Every function $\operatorname{Re} p(z)$, $p \in P$, has the Poisson representation by a unique positive measure ([3], p. 21–24, [8], p. 11–12)

$$\operatorname{Re} p(z) = \int_{-\pi}^{\pi} \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \tag{2}$$

where $d\mu(t) \geq 0$ and $\int_{-\pi}^{\pi} d\mu(t) = 1$; conversely, every function p holomorphic in $\mathbb{D}$ whose real part is given by (2), where $d\mu(t) \geq 0$ and $\int_{-\pi}^{\pi} d\mu(t) = 1$, and for which $\operatorname{Im} p(0) = 0$, is lying in P.

b) Let $d\mu(t) = f(t)\frac{dt}{2\pi} + d\sigma(t)$ be the Lebesgue decomposition of the representing measure μ with respect to the normalized Lebesgue measure $\frac{dt}{2\pi}$ on $\langle -\pi, \pi \rangle$, i.e. $\int_{-\pi}^{\pi} f(t)dt < \infty$, $f \geq 0$ almost everywhere (a.e.) on $\langle -\pi, \pi \rangle$ with respect to $\frac{dt}{2\pi}$ and $d\sigma$ is singular. Then $\operatorname{Re} p(z)$ has nontangential limits a.e. on $\langle -\pi, \pi \rangle$ and

$$(3) \qquad \operatorname{Re} p(e^{i\theta}) = f(e^{i\Theta}) \qquad \text{a. e. on } \langle -\pi, \pi \rangle$$

(see [8], Chapter 1, Th. 5.3).

In [5] the following subclass $\tilde{P}(B, b; \alpha)$, $0 \leq b < 1$, $b < B$, $0 < \alpha < 1$, of P was introduced: $p \in \tilde{P}(B, b; \alpha)$ if there exists an open arc $I_\alpha = I_\alpha(p)$ of $\mathbb{T}$ of length $2\pi\alpha$ such that

$$(4) \qquad \lim_{\substack{z \to z_0, \\ z \in \mathbb{D}}} \operatorname{Re} p(z) \geq B \qquad \text{for each } z_0 \in I_\alpha$$

and

$$(5) \qquad \lim_{\substack{z \to z_0, \\ z \in \mathbb{D}}} \operatorname{Re} p(z) \geq b \qquad \text{for each } z_0 \in \mathbb{T} \setminus \bar{I}_\alpha.$$

Among other results, the following properties of $\tilde{P}(B, b; \alpha)$ were proved in [5]: 1) a necessary and sufficient condition on the parameters B, b, α for $\tilde{P}(B, b; \alpha)$ to be nonvoid was given; 2) $\tilde{P}(B, b; \alpha)$ is compact in the topology given by the uniform convergence on compact subsets of $\mathbb{D}$; 3) $\tilde{P}(B, b; \alpha)$ is not convex.

In this paper we generalize all these results to the situation where arcs are replaced by closed measurable subsets of $\mathbb{T}$.

We start with the following reformulation of conditions (4), (5). The generalization is motivated by

Lemma 1. *Let $I_\alpha \subset \mathbb{T}$ be a given open arc and let $p \in P$. The following conditions are equivalent:*

(c) p fulfills conditions (4) and (5),

(d) p fulfills the conditions:

$$(6) \qquad \operatorname{Re} p(e^{i\theta}) \geq B \qquad \text{a. e. on } I_\alpha,$$

$$(7) \qquad \operatorname{Re} p(e^{i\theta}) \geq b \qquad \text{a. e. on } \mathbb{T} \setminus \bar{I}_\alpha.$$

Here $\operatorname{Re} p(e^{i\Theta})$ are nontangential limits of $\operatorname{Re} p$ which exist a.e. on $\mathbb{T}$ by (b).

Now, we are in a position to give our main definition.

Definition 1. Let $0 \le b < 1$, $b < B$, $0 < \alpha < 1$, be fixed real numbers and F a given closed subset of the unit circle $\mathbb{T}$ of Lebesgue measure $2\pi\alpha$. For each $\tau \in \langle -\pi, \pi)$, denote by $F_\tau = \{\xi \in \mathbb{T};\ e^{-i\tau}\xi \in F\}$ the set arising by rotation of F through the angle τ. Denote by $P(B, b, \alpha; F)$ the class of functions $p \in P$ satisfying the following conditions: there exists $\tau = \tau(p) \in \langle -\pi, \pi)$ such that

$$\operatorname{Re} p(e^{i\Theta}) \ge B \qquad \text{a.e. on} \quad F_\tau \tag{8}$$

and

$$\operatorname{Re} p(e^{i\Theta}) \ge b \qquad \text{a.e. on} \quad \mathbb{T}\backslash F_\tau. \tag{9}$$

It follows directly from Definition 1 that, for $B > 1$, the class $P(B, b, \alpha; F)$ does not contain the function $p_0(z) \equiv 1$, $z \in \mathbb{D}$. If $B \le 1$, then, clearly, $p_0 \in P(B, b, \alpha; F)$ for arbitrary admissible values of the parameters b, α and the set F.

In our further considerations, if it is not otherwise stated, we shall always assume that B, b, α, F and τ fulfill the conditions from Definition 1.

2. We have

Theorem 1. *If $P(B, b, \alpha; F) \neq \emptyset$, then*

$$1 \ge \alpha B + (1 - \alpha) b. \tag{10}$$

Proof. Let $p \in P(B, b, \alpha; F)$. So, there exists $\tau = \tau(p) \in \langle -\pi, \pi)$ such that (8) and (9) are fulfilled. Let $\omega(\cdot; F_\tau)$ be the harmonic measure of the set F_τ with respect to $\mathbb{D}$, i.e.

$$\omega(z; F_\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{F_\tau}(e^{it}) \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} dt \tag{11}$$

where $\chi_{\mathcal{A}}$ is the characteristic function of the set $\mathcal{A}$. Clearly, $0 < \omega(z; F_\tau) < 1$ in $\mathbb{D}$ and, by (3), $\omega(e^{it}; F_\tau) = 1$ a.e. on F_τ and $\omega(e^{it}; F_\tau) = 0$ on $\mathbb{T}\backslash F_\tau$. Put

$$u_\tau(z) = b + (B - b)\omega(z; F_\tau).$$

Then $u_\tau(z) = B$ a.e. on F_τ and $u_\tau(z) = b$ on $\mathbb{T}\backslash F_\tau$. Since, by (8) and (9), $\operatorname{Re}(p(e^{i\Theta}) - u_\tau(e^{i\Theta})) \ge 0$ a.e. on $\mathbb{T}$, we have, for each $z \in \mathbb{D}$, by (2) and (3),

$$\begin{aligned} \operatorname{Re} p(z) &= \int_{-\pi}^{\pi} \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \ge \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} p(e^{it}) \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} dt \\ &\ge \frac{1}{2\pi} \int_{F_\tau} B \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} dt + \frac{1}{2\pi} \int_{\mathbb{T}\backslash F_\tau} b \operatorname{Re} \frac{e^{it} + z}{e^{it} - z} dt = u_\tau(z), \end{aligned}$$

hence

$$\operatorname{Re} p(z) \ge b + (B - b)\omega(z; F_\tau), \qquad z \in \mathbb{D}. \tag{12}$$

For $z = 0$, we obtain, with respect to (1) and $\omega(0; F_\tau) = \alpha$, inequality (10).

Remark 1. Inequality (12) corresponds to the well-known two-constant theorem for bounded holomorphic functions ([1], p. 39).

Theorem 2. *Let condition (10) hold. Then, for each* $\tau \in \langle -\pi, \pi)$, *there exists a function* $p_{F_\tau} \in P(B, b, \alpha; F)$ *such that* $\operatorname{Re} p_{F_\tau}(e^{i\Theta}) = B$ *a.e. on* F_τ *and* $\operatorname{Re} p_{F_\tau}(e^{i\Theta}) = b$ *a.e. on* $\mathbb{T} \backslash F_\tau$.

Proof. Since the disc $\mathbb{D}$ is simply connected, therefore the function

$$h(z; F_\tau) = \omega(z; F_\tau) + i\omega^*(z; F_\tau), \tag{13}$$

where ω^*, $\omega^*(0) = 0$, is the harmonic conjugate of $\omega(z; F_\tau)$ is, by the monodromy principle, holomorphic in $\mathbb{D}$ for each $\tau \in \langle -\pi, \pi)$.

Let the equality in (10) hold. Then

$$p_{F_\tau}(z) = b + (B - b)h(z; F_\tau), \qquad z \in \mathbb{D}, \tag{14}$$

has the required property.

If $B\alpha + b(1 - \alpha) < 1$, then the function $\tilde{p}_{F_\tau}(z) = b + (B - b)h(z; F_\tau)$, $z \in \mathbb{D}$, fulfills (8) and (9) but does not belong to $P(B, b, \alpha; F)$ since $\tilde{p}_{F_\tau}(0) = B\alpha + b(1 - \alpha) < 1$. So, it is natural to achieve the required normalization by adding a proper multiple of $\frac{e^{i\gamma}+z}{e^{i\gamma}-z}$, γ real. Since on $\mathbb{T}$ we have $\operatorname{Re} \frac{e^{i\gamma}+z}{e^{i\gamma}-z} = 0$ a.e., therefore, clearly,

$$p_{F_\tau}(z) = \tilde{p}_{F_\tau}(z) + (1 - \eta)\frac{e^{i\gamma} + z}{e^{i\gamma} - z}, \qquad z \in \mathbb{D},$$

where

$$\eta = B\alpha + b(1 - \alpha), \tag{15}$$

is the required function.

Corollary 1. *The class* $P(B, b, \alpha; F)$ *is nonvoid if and only if inequality (10) holds. If in (10) the equality holds, then*

$$P(B, b, \alpha; F) = \{p_{F_\tau};\ \tau \in \langle -\pi, \pi)\}$$

where p_{F_τ} *is function (14).*

Theorem 3. *The class* $P(B, b, \alpha; F)$, *where* B, b, α *satisfy condition (10), is compact in the topology given by the uniform convergence on compact subsets of* $\mathbb{D}$.

Since $P(B, b, \alpha; F) \subset P$ and the class P is compact, it suffices to prove that $P(B, b, \alpha; F)$ is closed in P. In the proof of this fact one uses some properties of the harmonic measure $\omega(\cdot; F)$ of the set F and inequality (12). For details, see [7].

3. In this section we shall study sets $\mathbb{E}\ P(B, b, \alpha; F)$ and $\operatorname{supp} P(B, b, \alpha; F)$ of extreme points and support points of $P(B, b, \alpha; F)$, respectively ([9], p. 44, p. 91).

For this purpose, we denote, for a fixed $\tau \in \langle -\pi, \pi)$, $P(B, b, \alpha; F, \tau)$ - the set of all functions from $P(B, b, \alpha; F)$ satisfying (8) and (9) on F_τ. Clearly, $P(B, b, \alpha; F, \tau)$ is convex, compact and

$$P(B, b, \alpha; F) = \bigcup_{\tau \in \langle -\pi, \pi)} P(B, b, \alpha; F, \tau). \tag{16}$$

Proposition 1. *(i)* $P(B,b,\alpha;F,\tau) = \{p_{F_\tau} + (1-\eta)p;\ p \in P\}$ *where* p_{F_τ} *is function (14) and* η *is given by (15).*

(ii) For every τ*, the correspondence* $p \to p_{F_\tau} + (1-\eta)p$ *between the classes* P *and* $P(B,b,\alpha;F,\tau)$ *is one-to-one.*

(iii) $p \in P(B,b,\alpha;F,\tau_1)$ *if and only if* $\tilde{p}(z) = p(e^{i\tau}z) \in P(B,b,\alpha;F,\tau_1+\tau)$*.*

Now, denote by $\mathbb{E}(B,b,\alpha;F,\tau)$ the set of all $p(z;\gamma,F_\tau) \in P(B,b,\alpha;F,\tau)$ of the form

$$(17)\qquad p(z;\gamma,F_\tau) = b + (B-b)(z;F_\tau) + (1-\eta)\frac{e^{i\gamma}+z}{e^{i\gamma}-z}, \qquad \gamma - \text{real}, \quad z \in \mathbb{D},$$

and by $\mathcal{S}(B,b,\alpha;F,\tau)$ the set of all $s(z;F_\tau) \in P(B,b,\alpha;F,\tau)$ of the form

$$(18)\qquad s(z;F_\tau) = b + (B-b)h(z;F_\tau) + (1-\eta)\sum_{k=1}^{m}\lambda_k\frac{1+x_kz}{1-x_kz}, \qquad z \in \mathbb{D},$$

where $\lambda_k \geq 0$, $\sum_{k=1}^{m}\lambda_k = 1$ and $|x_k| = 1$; $m = 1,2,\ldots$.

From Proposition 1, the description of extreme points and support points of P, given in [9] (p. 48 and p. 94), and from (16) we immediately obtain

Corollary 2. *For arbitrary admissible* B,b,α,F,τ*, we have*

$$(19)\qquad \mathbb{E}P(B,b,\alpha;F,\tau) = \mathbb{E}(B,b,\alpha;F,\tau),$$

$$(20)\qquad \operatorname{supp} P(B,b,\alpha;F,\tau) = \mathcal{S}(B,b,\alpha;F,\tau),$$

$$(21)\qquad \mathbb{E}P(B,b,\alpha;F) \subset \bigcup_{\tau\in\langle-\pi,\pi)} \mathbb{E}(B,b,\alpha;F,\tau).$$

Theorem 4. *Let* $p \in P(B,b,\alpha;F)$ *have expansion (1) in* $\mathbb{D}$*. Then, for* $n = 1,2,\ldots,$

$$(22)\qquad |q_n| \leq 2\left[(B-b)\frac{1}{2\pi}\Big|\int_F e^{-int}dt\Big| + 1 - \eta\right].$$

This estimate is sharp and is attained only for functions (17) where $\gamma = -\frac{1}{n}\left(\arg\int_{F_\tau} e^{-int}dt + 2k\pi\right)$*,* $k \in \mathbb{Z}$ *(for* $\int_{F_\tau} = 0$*, we put* $\arg\int_{F_\tau} = 0$*).*

Proof. Since it is sufficient to verify estimate (22) for extreme points of $P(B,b,\alpha;F)$ (see e.g. [9], Th. 4.6, p. 45), by (21) we have only to make ourselves sure that the estimate holds for all functions of form (17) for all $\tau \in \langle-\pi,\pi)$ and is attained on some of them. So, we have only to write the Taylor expansion of functions (17). Since

$$\frac{e^{it}+z}{e^{it}-z} = 1 + 2\sum_{n=1}^{\infty} e^{-int}z^n, \qquad z \in \mathbb{D},$$

and the series converges uniformly in $\langle -\pi, \pi\rangle$ for $|z| < \rho < 1$, we can integrate term by term and obtain by elementary calculations (cf. (11), (13), (15), (17))

$$p(z;\gamma,F_\tau) = 1 + 2\sum_{n=1}^{\infty}\left[(B-b)\int_{F_\tau} e^{-int}dt + (1-\eta)e^{-in\gamma}\right] z^n,$$

so,

$$q_n = 2\left[(B-b)\frac{1}{2\pi}\int_{F_\tau} e^{-int}dt + (1-\eta)e^{-in\gamma}\right].$$

Denoting $\varphi_\tau = \arg\int_{F_\tau} e^{-int}dt$ if $\int_F e^{-int}dt \neq 0$ and putting $\varphi_\tau = 0$ in the opposite case, we have

$$q_n = 2\left[(B-b)\frac{1}{2\pi}\Big|\int_{F_\tau} e^{-int}dt\Big| + (1-\eta)e^{-i(n\gamma+\varphi_\tau)}\right] e^{i\varphi_\tau},$$

so,

$$\begin{aligned}|q_n| &= 2\Big|(B-b)\frac{1}{2\pi}\Big|\int_{F_\tau} e^{-int}dt\Big| + (1-\eta)e^{-i(n\gamma+\varphi_\tau)}\Big| \\ &= 2\Big|(B-b)\frac{1}{2\pi}\Big|\int_{F} e^{-int}dt\Big| + (1-\eta)e^{-i(n\gamma+\varphi_\tau)}\Big|.\end{aligned}$$

Since the first term of the sum is nonnegative, we obtain estimate (22).

Passing suitably to the limits, we obtain from (22) the well-known coefficient estimates in the classes P_b of Carathéodory (functions ([10]) of order b and in P ([2]).

4. Next, we consider

Definition 1. Let $0 \le b < 1$, $b < B$, $0 < \alpha < 1$, be fixed real numbers. Denote by $P(B,b,\alpha)$ the class of functions $p \in P$ such that there exists a closed subset F of $\mathbb{T}$ of Lebesgue measure $2\pi\alpha$ such that $p \in P(B,b,\alpha;F)$.

It follows directly from Definition 2 that

$$P(B,b,\alpha) = \bigcup_F P(B,b,\alpha;F) \tag{23}$$

where $F \subset \mathbb{T}$ satisfies the conditions mentioned above.

Our main theorem is the following

Theorem 5. *Let $p \in P(B,b,\alpha)$ have expansion (1) in $\mathbb{D}$. Then, for $n = 1,2,\dots$*

$$|q_n| \le 2\left[\frac{B-b}{\pi}\sin\alpha\pi + 1 - \eta\right]. \tag{24}$$

Estimate (24) is sharp and is attained only on the function $p^*(z) = p(\varepsilon z; F)$, $|\varepsilon| = 1$, $z \in \mathbb{D}$, where

$$F = F_n = \bigcup_{k=1}^{n} F_n^k, \qquad \textit{but} \quad F_n^k = \left\{z \in \mathbb{T};\ z = e^{\frac{2k\pi i}{n}}e^{i\rho},\quad \frac{-\alpha\pi}{n} \le \rho \le \frac{\alpha\pi}{n}\right\},$$

and so,

$$p(z;F)=b+\frac{B-b}{2\pi}\sum_{k=1}^{n}\int_{(-\alpha+2k)\pi/n}^{(\alpha+2k)\pi/n}\frac{e^{it}+z}{e^{it}-z}dt+(1-\eta)\frac{1+z}{1-z},\qquad z\in\mathbb{D}. \tag{25}$$

We give a rough sketch of the proof only; for details, see [7]. Let $p\in P(B,b,\alpha)$ have expansion (1) in $\mathbb{D}$. By (22), (23) and in view of the rotation invariance of the Lebesgue measure on $\mathbb{T}$, we easily obtain

$$|q_n|\le\frac{B-b}{\pi}Q_n+2(1-\eta),\qquad n=1,2,\dots,$$

where

$$Q_n=\sup_F\int_F\cos nt dt$$

and the supremum is taken over all closed subsets F of $\mathbb{T}$ having the Lebesgue measure $2\pi\alpha$. The following lemma is the clue to the proof of Theorem 5 (for the proof of the lemma, see [7]).

Lemma 2. *Let $a,b\in\mathbb{R}$ and let $E\subset\langle a,b\rangle$ be a measurable subset of the interval $\langle a,b\rangle$ and f a bounded nondecreasing function on $\langle a,b\rangle$. Then*

$$\int_a^{a+m(E)}f(t)dt\le\int_E f(t)dt\le\int_{b-m(E)}^{b}f(t)dt.$$

This lemma is used for the sets $F\cap I_k$ and $F\cap J_k$ where I_k and J_k are intervals, the function $\cos ine$ increases or decreases, respectively. After computations one obtains the following estimate of Q_n:

$$Q_n\le\sup\left\{\frac{2}{n}\sum_{n=1}^{\infty}\sin n\pi\alpha_k\right\}$$

where the supremum is taken over all systems $(\alpha_1,\dots,\alpha_n)$ such that $0\le\alpha_k\le\min(\frac{1}{n},\alpha)$ and $\sum_{k=1}^{n}\alpha_k=\alpha$. Finally, the concavity of $\sin x$ on $[0,\pi]$ gives result (29).

The form of the extremal functions is a consequence of the form of the set F shown in Theorem 5 and follows from formula (17). Since $P(B,b,\alpha;F_1\cup F_2)=P(B,b,\alpha;F_1)$ for an arbitrary closed set F_1 and an arbitrary set F_2 of Lebesgue measure 0 and such that $F_1\cup F_2$ is closed, therefore, for a fixed n, only function (25) is the function realizing the maximum of $|q_n|$ in the class $P(B,b,\alpha)$.

From Definition 1 and 2, Lemma 1 and Theorem 5 we get

Corollary 3. *Let $p\in\tilde{P}(B,b;\alpha)$ have expansion (1) in $\mathbb{D}$. Then, for $n=1,2,\dots,$*

$$|q_n|\le 2\left[\frac{B-b}{\pi}\sin\alpha\pi+1-\eta\right]. \tag{26}$$

Remark 2. Estimate (26) for $n=1$ is sharp. For $n=2,3,\dots,$ it is not sharp because function (25) belongs to the class $P(B,b,\alpha)$ but not to $\tilde{P}(B,b;\alpha)$. The sharp estimate in the class $\tilde{P}(B,b;\alpha)$ for $n=2,3,\dots$ is ([6])

$$|q_n|\le 2\left[\frac{B-b}{n\pi}|\sin n\alpha\pi|+1-\eta\right].$$

The estimate can also be obtained directly from (22).

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O oszacowaniach współczynników w pewnej klasie funkcji Carathéodory'ego o części rzeczywistej dodatniej

Streszczenie. Niech P oznacza znaną klasę funkcji $p(z) = 1 + q_1 z + \dots$ holomorficznych w kole jednostkowym $\mathbb{D}$ i spełniających warunek $\operatorname{Re} p(z) > 0$ w $\mathbb{D}$. Niech $0 \le b < 1$, $b < B$, $0 < \alpha < 1$ będą ustalonymi liczbami, zaś F danym domkniętym podzbiorem okręgu jednostkowego $\mathbb{T}$ o mierze Lebesgue'a $2\pi\alpha$. Dla każdego $\tau \in \langle -\pi, \pi)$ oznaczmy przez F_τ zbiór $\{\xi \in \mathbb{T};\ e^{-i\tau}\xi \in F\}$. Niech $\tilde{P}(B, b; \alpha)$ oznacza klasę funkcji $p \in P$ spełniających warunek: istnieje $\tau \in \langle -\pi, \pi)$ takie, że $\operatorname{Re} p(e^{i\Theta}) \ge B$ prawie wszędzie na F_τ oraz $\operatorname{Re} p(e^{i\Theta}) \ge b$ prawie wszędzie na $T \backslash F_\tau$.

W pracy zbadano podstawowe własności klasy $\tilde{P}(B, b; \alpha)$. Podano też oszacowania modułu współczynników w rodzinie $P(B, b, \alpha) = \bigcup_F \tilde{P}(B, b; \alpha)$. Pełny tekst pracy, w tym pominięte dowody twierdzeń, ukaże się w [7]. Otrzymane wyniki wchodzą w skład cyklu prac [4], [5], [6].

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