On Pinaki Mondal's generalization of the Kouchnirenko theorem

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→ $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$

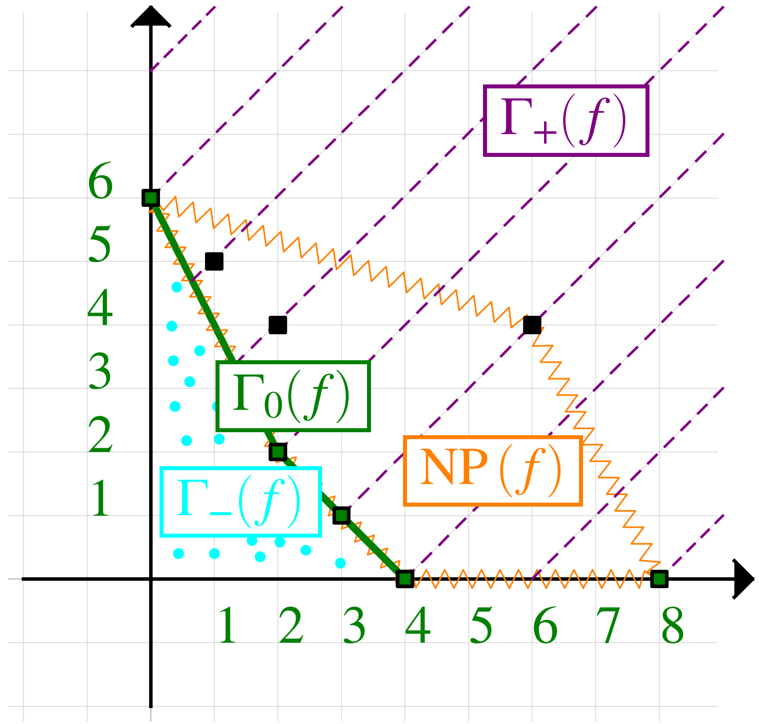
→ Let $n \geqslant 2$. For $f \in \mathbb{C}[[z_1, \ldots, z_n]]$, $f(0) = 0$, $f = \sum_{i \in \mathbb{N}_0^n} a_i\, z^i$, where $z^i = z_1^{i_1} \cdot \ldots \cdot z_n^{i_n}$ is the usual multi-index notation, we define:

- $\operatorname{Supp} f := \{i : a_i \neq 0\} \subset \mathbb{R}^n$, the support of f,
- $\Gamma_+(f) := \operatorname{conv}(\operatorname{Supp} f) + \mathbb{R}^n_{\geqslant 0}$, the Newton polyhedron of f
- $\Gamma_0(f)$:=the union of the compact faces of $\Gamma_+(f)$, called the Newton diagram of f,
- $\Gamma_-(f)$:=the union of all the segments joining the origin $0 \in \mathbb{R}^n$ with a point of $\Gamma_0(f)$.

Additionally, for f a *polynomial*, the set $\operatorname{NP}(f) := \operatorname{conv}(\operatorname{Supp} f)$ is called the Newton polygon of f.

→ $f \in \mathbb{C}[[z_1, \ldots, z_n]]$ is convenient if $\Gamma_0(f)$ touches every coordinate axis.

Let $f := (y^6 + x^2y^2 - 2x^3y + x^4) - 2xy^5 + x^2y^4 - 3x^6y^4 - \frac{2\pi i}{3}x^8$.



Let us recall

Definition

Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ *be a holomorphic germ. The* *Milnor number of* f *is*

$$\mu(f):=\dim_{\mathbb{C}}\frac{\mathbb{C}[[z]]}{(\nabla f)\,\mathbb{C}[[z]]}.$$

Problem. Can the value of the Milnor number of a germ f be computed combinatorically from the Newton diagram of f, if f is „non-degenerate" in some sense?

A positive answer to this question was given by A. G. Kouchnirenko.

Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a holomorphic germ, $f=\sum_{i\in\mathbb{N}_0^n} a_i z^i$. For any vector $v\in\mathbb{N}^n$ let S_v denote the face of $\Gamma_0(f)$ supported by v. We define the initial form of f with respect to v as $\mathrm{in}_v f := \sum_{i\in S_v} a_i z^i$.

→ We say that f is Kouchnirenko non-degenerate on a face S_v of $\Gamma_0(f)$ if the system

$$\{\nabla \mathrm{in}_v f = 0\} = \left\{\frac{\partial \mathrm{in}_v f}{\partial z_1} = \ldots = \frac{\partial \mathrm{in}_v f}{\partial z_n} = 0\right\}$$

has no solutions in $(\mathbb{C}^*)^n$, where $\mathbb{C}^* := \mathbb{C}\setminus\{0\}$.

→ We say that f is Kouchnirenko non-degenerate, if f is Kouchnirenko non-degenerate on every face S_v of its Newton diagram.

The basic version of Kouchnirenko theorem can be stated as follows:

Theorem 1 (Kouchnirenko '76)

If $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ is convenient, then:

1. $\mu(f)\geqslant\nu(f)$,
2. *if f is Kouchnirenko non-degenerate, then* $\mu(f)=\nu(f)<\infty$.

Moreover, the above non-degeneracy is „generic" in the space of all holomorphic germs g satisfying $\Gamma_0(g)=\Gamma_0(f)$.

Here, $\nu(f)$ is the combinatorial Newton number given by the formula

$$\nu(f):=\sum_{\mathcal{I}\subset\{1,\ldots,n\}}(-1)^{n-|\mathcal{I}|}\cdot|\mathcal{I}|!\cdot\mathrm{vol}_{|\mathcal{I}|}(\Gamma_-(f)\cap\mathbb{R}^{\mathcal{I}}),$$

where we put $\mathbb{K}^{\mathcal{I}}:=\{x\in\mathbb{K}^n: x_i=0 \text{ for } i\notin\mathcal{I}\}$, $\mathbb{K}=\mathbb{R}\vee\mathbb{C}$.

The previous result can be generalized to the case of non-convenient germs. This was remarked by Kouchnirenko and rigorously proved by Brzostowski and Oleksik.

Theorem 2 (B.–Oleksik)

If $f\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$ *and* $\nu(f)<\infty$, *then:*

1. $\mu(f)\geqslant\nu(f)$,
2. *if* f *is Kouchnirenko non-degenerate, then* $\mu(f)=\nu(f)$.

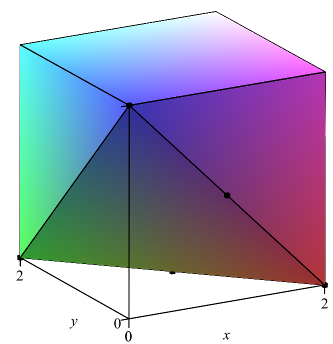
Moreover, the above non-degeneracy is „generic" in the space of all holomorphic germs g *satisfying* $\Gamma_0(g)=\Gamma_0(f)$.

Here, the definition of the Newton number is extended to the „non-convenient case" by the formula

$$\nu(f):=\sup_{k\in\mathbb{N}}\nu\Big(f+\sum_{1\leqslant i\leqslant n}z_i^k\Big).$$

→ In the 2–dimensional case Kouchnirenko's result is sharp, that is any germ f (convenient or not) satisfying $\mu(f) = \nu(f)$ has to be Kouchnirenko non-degenerate (Kouchnirenko '76, Płoski '99).

→ If $n \geqslant 3$ this is not the case. Here is the example provided by Kouchnirenko himself:
Consider $f := (x+y)^2 + xz + z^2$. Then f is Kouchnirenko degenerate with respect to the vector $v := (1,1,2)$: the system $\{\nabla \mathrm{in}_v f = 0\} = \{\nabla (x+y)^2 = 0\}$ possesses solutions in $(\mathbb{C}^*)^3$. Nevertheless, $\nu(f) = \mu(f) = 1$.



Question

How to make the Kouchnirenko theorem a sharp one?

→ Let $f_1, \ldots, f_m \in \mathbb{C}[[z_1, \ldots, z_n]]$, $m \geqslant n$. We say that $(f_1, \ldots, f_m)$ is Bernstein non-degenerate at 0 if for every $v \in \mathbb{N}^n$ the system

$$\{\mathrm{in}_v f_1 = \ldots = \mathrm{in}_v f_m = 0\}$$

doesn't have solutions in $(\mathbb{C}^*)^n$.

→ Let $m = n$. We say that $(f_1, \ldots, f_n)$ is Mondal non-degenerate if for all $\varnothing \neq \mathcal{I} \subset \{1, \ldots, n\}$ the tuple $(f_1|_{\mathbb{C}^{\mathcal{I}}}, \ldots, f_n|_{\mathbb{C}^{\mathcal{I}}})$ is Bernstein non-degenerate at 0.

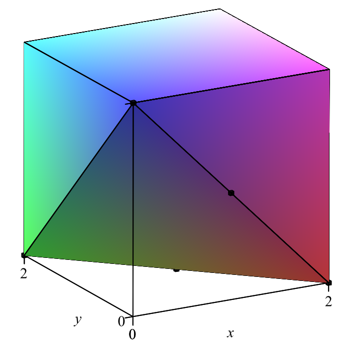
Mondal's non-degeneracy. Example

→ Mondal's non-degeneracy for gradients is in general <u>weaker</u> than Kouchnirenko's:

Consider again $f := (x+y)^2 + xz + z^2$. We have

$$\nabla f = (2x+2y+z, 2x+2y, x+2z).$$

As before take the vector $v := (1, 1, 2)$: the system $\left\{\mathrm{in}_v \frac{\partial f}{\partial x} = \mathrm{in}_v \frac{\partial f}{\partial y} = \mathrm{in}_v \frac{\partial f}{\partial z} = 0\right\} = \{2x+2y = x = 0\}$ has got *no* solutions in $(\mathbb{C}^*)^3$. The same thing can be checked for all other vectors $v \in \mathbb{N}^n$. Hence, ∇f is Bernstein at 0 (also Mondal) non-degenerate.



Since we are working over $\mathbb{C}$, Mondal's result can be stated as this:

Theorem 3 (Mondal)

Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$. Assume that $\nu(f)<\infty$. The f.s.a.e.:

1. *∇f is Mondal non-degenerate,*
2. *$\mu(f)=\nu(f)$.*

Commentary

Actually, P. Mondal assumes that $f\in\mathbb{C}[z_1,\ldots,z_n]$, not $f\in\mathbb{C}[[z_1,\ldots,z_n]]$. This seemingly weaker result implies the above one as follows:

→ if $g\in\mathbb{C}[z_1,\ldots,z_n]$ is a „partial sum” approximating f, so that in particular $\operatorname{ord}(f-g)=N$ and $N\gg 0$, then $\mu(g)=\mu(f)$,

→ ∇g and ∇f are both Mondal (non-)degenerate and $\nu(g)=\nu(f)$.

Lemma 1 (cf. Kouchnirenko '76)

Let $f \in \mathbb{C}[z_1, \ldots, z_n]$, $f(0) = 0$, and let z^k be a monomial, $k \in \mathbb{N}_0^n \setminus \{0\}$. Assume that $\mathrm{Supp}\, f \cup \{k\}$ is contained in a hyperplane of $\mathbb{R}^n$ with normal vector $l = (l_1, \ldots, l_n) \in \mathbb{N}^n$. Then for almost all choices of $s \in \mathbb{C}$ the function $g := f + s \cdot z^k$ is quasihomogeneous with weights l and the system $\{\nabla g = 0\}$ has no solutions in $(\mathbb{C}^*)^n$.

A direct consequence of the above lemma is:

Corollary 1

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and let z^k be a monomial, $k \in \mathbb{N}_0^n \setminus \{0\}$. Then for almost all choices of $s \in \mathbb{C}$ the function $g := f + s \cdot z^k$ is Kouchnirenko non-degenerate on all these faces of its Newton diagram which contain the point k. In particular, if f is Kouchnirenko non-degenerate, so is g, at least generically.

Proof of Lemma 1

→ Clearly, g is quasihomogeneous with weights l regardless of the value of $s \in \mathbb{C}$. By substituting $z \to (z_1^{l_1}, \ldots, z_n^{l_n})$, we may assume that g (and f) is homogeneous.

→ Put: $\kappa := (k_1, \ldots, k_{n-1}, 0)$, $d := \deg g$. Then $d > 0$, and we have $h(z) := g\left(\frac{z_1^d \cdot z_n}{z^\kappa}, \ldots, \frac{z_{n-1}^d \cdot z_n}{z^\kappa}, \frac{z_n}{z^\kappa}\right) = (\frac{z_n}{z^\kappa})^d \cdot g(z_1^d, \ldots, z_{n-1}^d, 1) = (\frac{z_n}{z^\kappa})^d \cdot (f(z_1^d, \ldots, z_{n-1}^d, 1) + s \cdot z_1^{d \cdot k_1} \cdot \ldots \cdot z_{n-1}^{d \cdot k_{n-1}}) = z_n^d \cdot (p(z_1, \ldots, z_{n-1}) + s)$,
where $h, p \in \mathbb{C}[z, z^{(-1,\ldots,-1)}]$ and p does not depend on z_n.

→ It is easy to see that the systems $\{\nabla g = 0\}$ and $\{\nabla h = 0\}$ are equivalent in $(\mathbb{C}^*)^n$. But $\{\nabla h = 0\} \underset{(\mathbb{C}^*)^n}{\Leftrightarrow} \{\frac{\partial p}{\partial z_1} = \ldots = \frac{\partial p}{\partial z_{n-1}} = p + s = 0\}$.

→ By Bertini-Sard theorem applied to p we get the assertion of the lemma.

For isolated singularities we can weaken Mondal's non-degeneracy condition:

Theorem 4

Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be an isolated singularity. The f.s.a.e.:

1. *∇f is Bernstein non-degenerate at 0,*
2. *∇f is Mondal non-degenerate,*
3. *$\mu(f)=\nu(f)$.*

→ We will prove this theorem indirectly, using P. Mondal's results.

→ By Theorem 3, and since „$(2)\Rightarrow(1)$” is trivial, we only need to show „$(1)\Rightarrow(3)$”.

Let us first note the following

Lemma 2

If $f_1, \ldots, f_n \in \mathbb{C}[[z_1, \ldots, z_n]]$, are all convenient, then $(f_1, \ldots, f_n)$ are Bernstein non-degenerate at 0 iff they are Mondal non-degenerate.

Proof of the lemma

„⇒". Take any $\varnothing \neq \mathcal{I} \subset \{1, \ldots, n\}$. Without loss of generality, we may assume that $\mathcal{I} = \{1, \ldots, p\}$, $p < n$. Take any $v = (v_1, \ldots, v_p) \in \mathbb{N}^p$. Put $v_\infty := (v_1, \ldots, v_p, N, \ldots, N) \in \mathbb{N}^n$, where $N \gg 0$. By assumption, $f_i|_{\mathbb{C}^\mathcal{I}} = f_i(z_1, \ldots, z_p, 0, \ldots, 0) \neq 0$ $(i = 1, \ldots, n)$. Hence, $\mathrm{in}_v(f_i|_{\mathbb{C}^\mathcal{I}}) = \mathrm{in}_{v_\infty} f_i$, for $i = 1, \ldots, n$. This means that the system $\{\mathrm{in}_v(f_i|_{\mathbb{C}^\mathcal{I}}) = 0\}_{1 \leqslant i \leqslant n}$ has no solutions in $(\mathbb{C}^*)^n$ and – consequently – no solutions in $(\mathbb{C}^*)^\mathcal{I}$.

„⇐". Trivial.

Proof of implication „$(1) \Rightarrow (3)$” of the theorem

- → Assume that ∇f is Bernstein non-degenerate at 0. Consider $g(z) := f(z) + a(z)$, where $a(z)$ is a generic enough form of degree $N \gg 0$. Then $\mu(g) = \mu(f) < \infty$. By Corollary 1, ∇g is Bernstein non-degenerate at 0.
- → Since we may assume that all the $\frac{\partial g}{\partial z_i}$ are convenient, Lemma 2 asserts that ∇g is Mondal non-degenerate. For the same reason, $\nu(g) < \infty$. Hence, Theorem 3 gives the equality $\mu(g) = \nu(g)$. Consequently, $\mu(f) = \nu(g)$.
- → On the other hand, Theorem 2 allows us to find an isolated singularity $\bar{f}$ which is Kouchnirenko non-degenerate and $\Gamma_0(\bar{f}) = \Gamma_0(f)$. Defining $\bar{g}$ similarly as above, we get $\nu(f) = \nu(\bar{f}) = \mu(\bar{f}) = \nu(\bar{g}) = \nu(g)$.
- → Summing up, $\mu(f) = \nu(f)$.

P. Mondal also gives a criterion for a map-germ to have its intersection multiplicity at 0 computable using a combinatorial quantity.

Theorem 5 (Mondal)

Let $f = (f_1, \ldots, f_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. *Assume that* $(\Gamma_1(f_1), \ldots, \Gamma_n(f_n))_0 < \infty$.
The f.s.a.e.:

1. $(f_1, \ldots, f_n)$ *is Mondal non-degenerate,*
2. $(f_1, \ldots, f_n)_0 = (\Gamma_1(f_1), \ldots, \Gamma_n(f_n))_0$.

→ Here, $(\Gamma_1(f_1), \ldots, \Gamma_n(f_n))_0$ is a notation for the „generic” (=minimal) value of the intersection multiplicity for map-germs with the same n-tuple of Newton diagrams as f's one.

→ Moreover, Mondal gives (a rather complicated) combinatorial formula for $(\Gamma_1(f_1), \ldots, \Gamma_n(f_n))_0$.

It turns out that under the condition that all f_i are convenient, one part of Mondal's theorem has already been proved (see the book of Aizenberg&Yuzhakov, Thms. 22.9, 22.10):

Theorem 6

Let $f = (f_1, \dots, f_n) \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. If all the f_i are convenient, then:

1. $(f_1, \dots, f_n)_0 \geqslant (\Gamma_1(f_1), \dots, \Gamma_n(f_n))_0$,
2. *if* $(f_1, \dots, f_n)$ *is Bernstein non-degenerate at* 0*, then* $(f_1, \dots, f_n)_0 = (\Gamma_1(f_1), \dots, \Gamma_n(f_n))_0 < \infty$.

Moreover, the above non-degeneracy is „generic" in an appropriate sense.

Remark

Actually, the statement of the above theorem given by Aizenberg and Yuzhakov is incorrect, because it doesn't guarantee that the f_i are convenient and the proof requires this (P. Mondal, personal communication to Prof. Krasiński).

Example (Mondal, personal communication to Prof. Krasiński)

→ Consider $f := x+y+z$, $g := x+y+2z+x^2$, $h := z\cdot(x+2y+3z)$.

→ It is easy to see that the system is Bernstein non-degenerate at 0.

→ However, if you restrict the system to the (x, y)-plane you get $\{x+y = x+y+x^2 = 0\}$ and this system is Bernstein degenerate at 0 with respect to the vector $v := (1,1)$. Hence, (f,g,h) is Mondal degenerate.

→ Correspondingly, $(f,g,h)_0 = 3$ but $(\Gamma_0(f), \Gamma_0(g), \Gamma_0(h))_0 = 2$.

→ Any „convenientation" of the system, e.g. $\bar{f} := f$, $\bar{g} := g$, $\bar{h} := h + a\,x^k + b\,y^l$, $l > k \geqslant 4$, leads to a system which is always Bernstein degenerate at 0, as can be seen by considering the vector $v := (1,1,k-1)$: $\mathrm{in}_v \bar{h} = z\cdot(x+2y) + a\,x^k$.

→ Hence, an analogue of Lemma 1 is not valid in general and one cannot repeat the reasoning from the proof of Theorem 4 in the case of arbitrary systems.

Dziękuję za uwagę!